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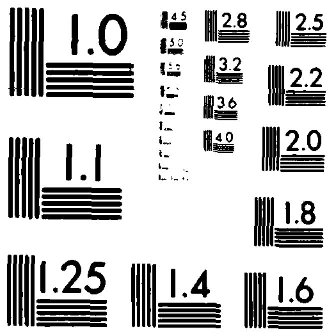

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WAVE NUMBER SHOCKS FOR THE TAIL OF KORTEWEG-DEVRIES

SOLITARY WAVES IN SLOWLY VARYING MEDIA

A Dissertation Presented to the Graduate Faculty of

Dedman College

of

Southern Methodist University

in

Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

in

Mathematical Sciences

by

Darrell Emory Allgaier

(B.S., University of Central Florida, 1975)

(M.S., University of Central Florida, 1979)

April 7, 1986

WAVE NUMBER SHOCKS FOR THE TAIL OF KORTEWEG-DEVRIES

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Wave Number Shocks for the Tail of Korteweg-deVries Solitary Waves in Slowly Varying Media

Advisor: Professor Richard Haberman

Doctor of Philosophy degree conferred May 17, 1986

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Asymptotic solutions for the nonlinear, nonhomogeneous, Korteweg-deVries (KdV) partial differential equation (pde) with slowly varying coefficients are not in general uniformly valid. A uniform asymptotic expansion is obtained by finding separate expansions for different regions and matching. A KdV solitary wave propagating in slowly varying media is examined. Quasi-stationarity for the core reduces the problem to solving ordinary differential equations for that region. However, in the leading tail region, hyperbolic pde's must be solved to determine the amplitude and phase. The method of characteristics predicts triple valuedness after a caustic (penumbral or cusped) develops. Singular perturbation methods show the solution near first focusing satisfies the diffusion equation and involves either an incomplete Airy-type integral or an exponential integral similar to the Pearcey integral. Laplace's method shows that the critical points of the exponential phase satisfy the fundamental folding equation. A linear multi-phase solution is determined which does not become triple valued (break). Instead, a wave number shock develops, which separates two different solitary wave

tails, and travels at the shock velocity predicted by conservation of waves. Thus, a unique uniform leading tail solution is obtained corresponding to a specified moving core (the problem is shown to be well-posed).

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ACKNOWLEDGMENTS

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I cannot express my gratitude for the love, understanding and support of my wife, Lyla. I dedicate this thesis to her and to our children, John, Jenniffer, Sarah, Diana, and Susan, who fill our lives with joy.

CHAPTER I

INTRODUCTION

1.1 History

The objective of this thesis is to add to our understanding of a physical phenomenon which was first scientifically observed over 150 years ago by J. Scott Russell. The incident in 1834 is best described by Russell himself:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped--not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined head of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height." [30]

Russell's life was dramatically changed by this single event, and so was our understanding of wave motion to change as the investigation began. He found no mathematical theory available which predicted such a phenomenon and proceeded to study it experimentally. At the time he could hardly have imagined the controversy that would arise between him and Airy over the existence of such a permanent waveform, much less have appreciated the bountiful applications in science that would arise and the number of great minds that would apply themselves to broadening our

understanding of this phenomenon! It was nearly fifty years before the controversy was put to rest by such men as Boussinesq, Rayleigh, and Korteweg and deVries. They demonstrated the existence of a permanent solitary wave for nonlinear partial differential equations of shallow water theory. There was little scientific activity for another half century until other physical applications were shown to be governed by similar mathematics. Then a variety of situations were shown to give rise to nonlinear dispersive partial differential equations (collision-free hydromagnetic waves by Gardner and Morikawa and a nonlinear meson field theory by Perring and Skyrme). Fermi, Pasta and Ulam's study of heat conductivity in solids then motivated Zabusky and Kruskal's important numerical study of solitary waves, coined "solitons" by them because they observed particle-like interactions. In order to understand these interactions, a theoretical investigation was undertaken resulting in one of the most significant developments in applied mathematics and mathematical physics in recent years. Gardner, Greene, Kruskal and Miura related inverse scattering for the Schrodinger eigenvalue problem to the nonlinear Korteweg-deVries (KdV) equation. In this way they obtained solitons and multiply-interacting solitons. Shortly thereafter Lax showed that other nonlinear partial differential equations could be analyzed in this way. A series of interesting physical problems was studied (self-induced transparency by McCall and Hahn collaborated by Lamb, and the nonlinear Schrodinger equation by Zakharov and Shabat, both important examples since they introduced a different eigenvalue problem). Finally, Ablowitz, Kaup, Newell and Segur showed how the inverse scattering transform (IST) could be used to

solve a wide class of nonlinear evolution equations. Thus Russell's discovery of the solitary wave has led to much research and no doubt will lead to much much more.

1.2 Motivation

A search of the literature shows much activity and excitement in the area of solitary waves and solitons once Gardner, Greene, Kruskal and Miura [4] demonstrated their method for obtaining exact solutions to the nonlinear KdV equation. Their method of direct and inverse scattering, which views the eigenfunctions as the transmitted portion of a wave coming in from infinity and being scattered by the initial conditions, was formalized by Ablowitz, Kaup, Newell and Segur in 1974 [2]. This formalization demonstrated permanent solitary wave solutions for a large class of problems. The technique, however, requires that the problem be integrable, which is not the case for many problems which involve slow variations. It is precisely these type problems which arose in many physical applications. Researchers thus began using numerical techniques to gain understanding, and singular perturbation methods to analytically demonstrate asymptotic behavior. In the late 1970's several independent studies showed the formation of a "shelf" behind a slowly varying KdV solitary wave. Kaup and Newell [14] and Knickerbocker and Newell [19] used perturbations on the inverse scattering solution, while Ko and Kuehl [16,17], Karpman and Maslov [13], and Grimshaw [5] used direct perturbation methods. For the case of a KdV soliton propagating to the right through a region of slowly changing depth, Kaup and Newell write:

> "if a soliton propagates through a region of decreasing depth to a new constant level h_1 at $x = x_1$, the solution in the region to the right of $x = x_1$ will consist of (1) the original soliton, whose height η^2 has adjusted in the transition region so as to be just the correct height for the new depth plus (2) a number of secondary solitons produced by the arrival at $x = x_1$ of the elevated shelf (which may be treated approximately as a rectangular well potential of width $L \simeq (\overline{x} - x_0)$ and average height $H \approx -L/3\eta$; the number of solitons will be proportional to $\sqrt{HL}$) and (3) some left-over radiation." [14]

It is this case of a solitary wave propagating on the surface of water of variable depth which Grimshaw [5] considers using a multiple scale approach. He shows that the amplitude and phase of slowly varying solitary waves are determined by hyperbolic partial differential equations. In his paper he points out for the region in front of the wave:

> "A caustic develops...Beyond the caustic f_0 becomes multivalued, and it is tempting to conjecture that the breakdown is associated with the formation of further solitary waves." [5]

It is here that we pick up the study. The development of a shelf behind a slowly varying solitary wave has been extensively studied (see above), but the situation in which characteristics cross in front of the wave has not been analytically explained. The primary impetus for the study is thus to explain analytically what the development of a caustic in front of the wave means in terms of the physical solution. Is Grimshaw's conjecture correct?

1.3 Results

First we show that the problem separates asymptotically into two parts. We call the two parts of the slowly varying solitary wave the

"core", or the region where the mass is concentrated, and the "leading tail", or exponentially decaying region in front of the core with respect to its motion. Our first results came by making some simplifying assumptions. Looking first to the leading tail we linearize the problem and then consider the special case of constant coefficients, with slow variations introduced through initial conditions only. We use the methods of characteristics, multiple scales, matched asymptotic expansions and asymptotics of exponential integrals. The results are quite interesting. Further solitary waves are not created in the leading tail as conjectured by Grimshaw. Instead, we find that the triple valuedness predicted by the method of characteristics actually leads to a "wave number shock", or a relatively rapid transition from one slowly varying wave tail to another. Details of these results are given in Haberman and Allgaier [8]. The idea of wave number shocks was first discussed by Howard and Kopell [10] in their work on slowly varying reaction-diffusion equations. Also Haberman and Sun [9] used similar asymptotic calculations in their work on slowly varying oscillatory dispersive waves.

The remaining chapters of this thesis show how the properties of the leading tail relate to the structure of the solitary wave core. Futhermore, the most general cases of arbitrary slowly varying co- efficients and arbitrary slowly varying initial conditions for both the core and the leading tail are solved.

We show, in fact, Grimshaw's conjecture (see Section 1.2) is partially correct, that under the right conditions characteristics for the leading tail will cross and lead to the formation of a new type of solitary wave tail. We show that the set of different conditions, which

lead to a wave number shock in the leading tail of a solitary wave, contains the particular conditions which Smyth [2] demonstrates lead to the formation of further solitary waves in the region behind the core.

Finally, we find some very interesting facts concerning the formation of caustics. The specific slowly varying coefficients and slowly varying initial conditions are what determine the type of caustic which arises. In general, a cusped caustic (so named because of its shape, see Figure 4.1) occurs when the crossing of characteristics is approximated by a complete cubic equation. This can occur both for characteristics generated by the moving core and for characteristics generated by initial conditions in the leading tail. On the other hand, a penumbral caustic (here penumbral means partial illumination resulting from the cut-off of some rays or characteristics as in an eclipse) occurs when the crossing of characteristics is approximated by an incomplete (or cut-off) quadratic or cubic equation. This only occurs when the characteristic eminating from the initial moving core catches up to the characteristic from the nearest initial leading tail (see Figures 3.1 and 3.3). For much more detail on the classification of caustics we refer the reader to the work by Kravtsov and Orlov [20].

1.4 Synopsis

In Chapter III we derive two asymptotic eq·ations, valid in different regions, which describe the behavior of Korteweg-deVries solitary waves in slowly varying media. In the core we use quasi-stationarity to obtain the asymptotic solution. However, in the leading tail quasi-stationarity is not appropriate, thus we use the more

standard multiple scale approach.

We next show, in Chapter III, that the problem of a KdV solitary wave propagating through slowly varying media is similar in some respects to the accelerating piston problem in gas dynamics. The method of characteristics shows that, for certain physically interesting problems, characteristics cross in the leading tail causing a caustic to form which predicts a region of triple valuedness. By examining the local behavior near the point of first crossing, we find it useful to introduce a similarity variable. The breakdown of the asymptotic expansion leads to rescaling. The new scales show that the leading order solution near the first crossing of characteristics satisfies the diffusion equation, even though the wave could be purely dispersive without dissipation. Using Laplace's method for the asymptotics of exponential integrals, we show that the solution does not become triple valued (break), but instead rapid transitions occur from one exponentially decaying tail to another, a wave number shock.

In Chapter III we show that a penumbral caustic is generated when the first crossing of characteristics occurs along the characteristic eminating from the initial core position. Then, in Chapter IV, we show for the other two cases (first crossing along a characteristic eminating from the moving core at a later position or eminating from the initial leading tail) that a cusped caustic is generated. In all three of these cases we show that the predicted triple valuedness actually corresponds to the propagation of a wave number shock in the leading tail.

CHAPTER II

DERIVATION OF THE EQUATIONS

2.1 Introduction

In this chapter we seek partial differential equations (pde's), using perturbation techniques, which describe the behavior of slowly varying solitary waves and solitons [1] moving in inhomogeneous media.

Since the pioneering work of Gardner, Greene, Kruskal and Miura [4] in 1967, it has become well known that the constant coefficient Korteweg-deVries (KdV) equation has a family of exact N-soliton solutions. These solutions can be obtained using the inverse scattering transform technique [1,29] which was derived for exactly integrable equations. The variable coefficient KdV equation, which in general is not exactly integrable, describes many interesting physical phenomena [6, 11, 12, 17, 21, 25, 26] and has been discussed extensively in the literature. Grimshaw [5] and Johnson [6] consider the case of a solitary wave propagating on the surface of water of variable depth. We make a minor generalization to their work by allowing the variable coefficients to depend on space as well as time.

Using a multiple scale approach, Grimshaw [5] showed that the asymptotic expansion breaks down as one gets too far from the peak or center of the wave. We thus consider two cases, first the "core" or thin region near the center, and then the "leading tail" or semi-

infinite region on the forward side of the wave.

2.2 The Core

We consider the nonhomogeneous, variable coefficient KdV equation

$$u_t + \nu u u_x + \lambda u_{xxx} = -\varepsilon \gamma u \ , \tag{2.1}$$

with a solitary wave initial condition. In this equation $0 < \varepsilon \ll 1$ is a small positive parameter and , and are slowly varying functions of $X = \varepsilon x$ and $T = \varepsilon t$. Using the standard multiple scale approach, we take the fast phase variable to be

$$\eta = \theta(X,T)/\varepsilon, \tag{2.2}$$

and assume

$$u(x,t) = U(X,T,\eta). \tag{2.3}$$

The usual chain rule yields

$$u_t = \varepsilon(U_t + U_\eta \eta_T) \ ,$$

$$u_x = \varepsilon(U_X + U_\eta \eta_X) \ ,$$

and

$$u_{xxx} = \varepsilon^3[U_{XXX} + U_\eta \eta_{XXX} + 3U_{X\eta}\eta_{XX} + 3U_{XX\eta}\eta_X$$

$$+ 3U_{\eta\eta}\eta_{XX}\eta_X + 3U_{X\eta\eta}(\eta_X)^2 + U_{\eta\eta\eta}(\eta_X)^3] \ .$$

We define a slowly varying wave number, k, and a slowly varying frequency, w, as follows:

$$\eta_x = \theta_X \equiv k, \tag{2.4.a}$$

$$\eta_t = \theta_T = -w \ , \tag{2.4.b}$$

which yields conservation of waves

$$k_T + w_X = 0 \ . \tag{2.5}$$

Using (2.3) and (2.4), equation (2.1) becomes

$$-wU_\eta + \nu k U U_\eta + \lambda k^3 U_{\eta\eta\eta} =$$

$$-\varepsilon(U_T + \nu U U_X + 3\lambda k k_X U_{\eta\eta} + 3\lambda k_X U U_{X\eta\eta} + \gamma U)$$

$$-\varepsilon^2(\lambda k_{XX} U_\eta + 3\lambda k_X U_{X\eta} + 3\lambda k U_{XX\eta}) - \varepsilon^3(\lambda U_{XXX}) . \qquad (2.6)$$

To solve this equation, we expand U, k and w in powers of ε

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots ,$$

$$k = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots ,$$

and

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots .$$

Thus, to leading order, equation (2.6) becomes

$$-w_0 U_{0\eta} + \nu k_0 U_0 U_{0\eta} + \lambda k_0^3 U_{0\eta\eta\eta} = 0 . \qquad (2.7)$$

The well known solitary wave solution for equation (2.7) is

$$U_0 = A \operatorname{sech}^2 \beta\eta , \qquad (2.8)$$

$$A = \frac{3w_0}{\nu k_0} ,$$

$$\beta = \left(\frac{w_0}{4\lambda k_0^3}\right)^{1/2} ,$$

which agrees with Grimshaw [5]. Continuing to higher orders shows that the expansion for U breaks down when $|\eta| > O(1)$. We thus define the core by $\eta = O(1)$ (assuming $\beta = O(1)$), since otherwise U_0 is exponentially small. The center of the core is $\eta = 0$. We take the position of this center to be an unknown function of time

$$X_c = X_c(T) = X_{c_0}(T) + \varepsilon X_{c_1}(T). \qquad (2.9)$$

Thus, at any given time, we find from equation (2.4.a) that the fast phase expanded around its center is given by

$$\eta \approx k_0(X_c,T)[X-X_c]/\varepsilon, \tag{2.10}$$

to leading order. Thus the core occurs where $X-X_c = O(\varepsilon)$ (where (2.10) is valid), or to within order ε of the path $X = X_{c_0}(T)$. Now using "^" to denote evaluation at $X = X_{c_0}$, we find that the leading order solution (2.8) in the core, is given by

$$U_0 \approx \frac{3\hat{w}_0}{\hat{\nu}\hat{k}_0}\,\mathrm{sech}^2\left[\left(\frac{\hat{w}_0}{4\hat{\lambda}\hat{k}_0}\right)^{1/2}\frac{X-X_c}{\varepsilon}\right]. \tag{2.11}$$

This is a quasi-stationary solution [18]. We redefine the fast phase to be

$$\eta = \frac{X-X_c}{\varepsilon}. \tag{2.12}$$

From (2.4) this choice of phase yields

$$k \equiv 1, \tag{2.13.a}$$

and

$$w \equiv X_c'(T), \tag{2.13.b}$$

automatically satisfying (2.5). We will shortly determine the unknown $w(T)$. In terms of it the center of the core is

$$X_c(T) = X_c(0) + \int_0^T w(\bar{T})d\bar{T}, \tag{2.13.c}$$

where $X_c(0)$ is the initial center. It is thus easier to directly solve (2.1) in the core using a quasi-stationary approach with two scales, rather than the standard slowly varying approach with three scales:

$$u(x,t) = V(\eta,T). \tag{2.14}$$

We expand the coefficients around $X = X_{c_0}$ to obtain

$$-wV_\eta + \hat{\nu}VV + \hat{\lambda}V = -\varepsilon[V_T + \hat{\nu}_X(\eta + X_{c_1})VV_\eta + \hat{\lambda}_X(\eta + X_{c_1})V_{\eta\eta\eta} + \hat{\gamma}V] + O(\varepsilon^2). \tag{2.15}$$

Thus the leading order equation is

$$-w_0 V_{0\eta} + \hat{\nu} V_0 V_{0\eta} + \hat{\lambda} V_{0\eta\eta\eta} = 0 \ , \tag{2.16}$$

with the solution

$$V_0 = \frac{3w_0}{\hat{\nu}} \operatorname{sech}^2 [(\frac{w_0}{4\hat{\lambda}})^{1/2} \frac{X - \int_0^T w(\bar{T}) d\bar{T}}{\varepsilon}] \ . \tag{2.17}$$

Here we have taken the center of the core to be initially at $X = 0$. Note that the sign of w_0 equals the sign of $\hat{\lambda}$ and hence equations (2.13.b) and (2.17) show that the core must move in the direction of the sign of $\hat{\lambda}$. Continuing to the next order we obtain

$$-w_0 V_{1\eta} + \hat{\nu}(V_0 V_1)_\eta + \hat{\lambda} V_{1\eta\eta\eta} = w_1 V_{0\eta}$$

$$- (V_{0_T} + \hat{\nu}_X (\eta + X_{c_1}) V_0 V_{0\eta} + \hat{\lambda}_X (\eta + X_{c_1}) V_{0\eta\eta\eta} + \hat{\lambda} V_0) \ . \tag{2.18}$$

We now define the linear operator, L, and its adjoint, L^A, as did Kodama and Ablowitz [18]:

$$Ly \equiv -w_0 y_\eta + \hat{\nu}(V_0 y)_\eta + \hat{\lambda} y_{\eta\eta\eta},$$

$$L^A y \equiv w_0 y_\eta - \hat{\nu} V_0 y_\eta - \hat{\lambda} y_{\eta\eta\eta},$$

so that

$$\int_{-\infty}^{\infty} (V_0 L V_1 - V_1 L^A V_0) d\eta =$$

$$[+\hat{\lambda}(V_0 V_{1\eta\eta} - V_{0\eta} V_{1\eta} + V_{0\eta\eta} V_1) + \hat{\nu} V_0^2 V_1 - w_0 V_0 V_1]_{-\infty}^{\infty} \ .$$

Thus V_1 and its first two partial derivatives with respect to η will be well behaved if the following capatibility condition is satisfied

$$\int_{-\infty}^{\infty} (V_0 V_{0_T} + \hat{\nu}_X (\eta + X_{c_1}) V_0^2 V_{0\eta} + \hat{\lambda}_X (\eta + X_{c_1}) V_0^2 V_{0\eta\eta\eta} + \hat{\gamma} V_0) d\eta = 0 \ . \tag{2.19}$$

Using equation (2.17) to evaluate (2.19) we obtain

$$\frac{d}{dT} (\frac{\hat{\lambda}^{1/2} w_0^{3/2}}{\hat{\nu}^2}) = (\frac{96}{5} \frac{\hat{\nu}_X}{\hat{\nu}} w_0 - 2\hat{\gamma})(\frac{\hat{\lambda}^{1/2} w_0^{3/2}}{\hat{\nu}^2}) \ , \tag{2.20}$$

which can be solved for $w_0(T)$ from an initial condition $w(0)$ (we assume $w(0)$ independent of ε so that $w_0(0) = w(0)$). Thus, the leading order core solution, (2.17), is completely determined using (2.13.b).

Equation (2.20) appears to be a first order ordinary differential equation (ode) with variable coefficients. However, in general the position of the core depends on the unknown w_0 via equation (2.13.c). We see that $w_0 = X'_{c_0}$ and $w'_0 = X''_{c_0}$. Thus (2.20) is actually a second order ode with variable coefficients. To solve it we must know the initial conditions $X_c(0)$ and $X'_c(0)$. If ν and λ are functions of T only, then (2.20) has the solution

$$w_0(T) = w(0)\left[\frac{\lambda(0)\nu^4(T)}{\nu^4(0)\lambda(T)}\right]^{1/3} \exp\left[-\frac{4}{3}\int_0^T \gamma(\bar{T})d\bar{T}\right] ,$$

as shown by Ablowitz and Segur [1].

We note here that if the perturbation analysis using equation (2.3) is carried forward without assuming quasi-stationarity, we can obtain a hyperbolic system of pde's, in X and T, for the wave number and amplitude, as did Grimshaw [5]. We must be careful to realize that these pde's are only valid in the core; X is in a very thin region near the center. From equation (2.10) we see that

$$X \approx X_c(T) + O(\varepsilon).$$

In this thin region, X is, to leading order, a function of T, thus reducing the pde's to ode's. If we fail to show proper care, and use the method of characteristics to solve the hyperbolic system, after much tedious algebra and some complicated matching, we find that the solution fails to focus in the core region. This is the same result we have obtained using quasi-stationarity!

2.3 The Leading Tail

The semi-infinite regions on either side of the core we refer to as tails. The tail on the side of the core in the direction of its motion we call the leading tail. We call the other one the trailing tail. There is much discussion in the literature concerning the trailing tail of slowly varying solitary waves and the development of a "shelf-like" structure [13, 16, 17, 18, 19, 24, 25, 28]. There appears to be very little discussion concerning the crossing of characteristics, which leads to the formation of a caustic, for the leading tail [5]. It is this problem we are concerned with here.

In the leading tail the quasi-stationarity assumption is not valid, since the tail is not localized in space. However, due to the decay as of the solution, (2.17), the nonlinear term in equation (2.1) is exponentially small. Thus, as is justified by the method of matched asymptotic expansions, we seek a slowly varying solution to

$$u_t + \lambda u_{xxx} = -\varepsilon\gamma u \; . \qquad (2.21)$$

We expect the leading tail of the solitary wave to be exponentially decaying. A steady traveling decaying wave solution would be

$$Ae^{KX-\Omega T},$$

similar in mathematical form to an oscillatory dispersive wave. However, (2.21) has slow variations including a slow decay or growth introduced by the term with . Thus we use the multiple scale approach and introduce terminology similar to that used f r oscillatory dispersive waves: the fast phase, H, slowly varying wave number, K, and slowly varying frequency, . They satisfy the conditions

$$H_x = \theta(X,T)/\varepsilon, \tag{2.22}$$

$$H_x = \theta_X \equiv K , \tag{2.23.a}$$

and

$$H_t = \theta_T \equiv -\Omega , \tag{2.23.b}$$

so that conservation of waves

$$K_T + \Omega_X = 0 , \tag{2.24}$$

also holds in the leading tail. Using the chain rule on

$$u(x,t) = U(X,T,H),$$

equation (2.21) becomes

$$-\Omega U_H + \lambda K^3 U_{HHH} = -\varepsilon(U_T + 3\lambda K K_X U_{HH} + 3\lambda K^2 U_{XHH} + \gamma U)$$

$$-\varepsilon^2(\lambda K_{XX} U_H + 3\lambda K_X U_{XH} + 3\lambda K U_{XXH}) - \varepsilon^3(\lambda U_{XXX}) . \tag{2.25}$$

The leading order equation

$$-\Omega_0 U_{0H} + \lambda K_0^3 U_{0HHH} = 0 , \tag{2.26}$$

has the decaying solution (as $H \to \infty$, assuming $\lambda>0$ or a right going wave)

$$U_0 = A(X,T)\exp[-(\frac{\Omega_0}{\lambda K_0^3})^{1/2} H] . \tag{2.27}$$

The next order equation is

$$-\Omega_0 U_{1H} + \lambda K_0^3 U_{1HHH} = \Omega_1 U_{0H} - 3\lambda K_0^2 K_1 U_{0HHH} - U_{0T}$$

$$-3\lambda K_0 K_{0X} U_{0HH} - 3\lambda K_0^2 U_{0XHH} - \gamma U_0 . \tag{2.28}$$

The right hand side of this equation produces secular terms for U_1 (terms which, for large H, cause the asymptotic expansion to break down) unless

$$\Omega_0 = \lambda K_0^3 , \tag{2.29}$$

and

$$A_T + 3\lambda K_0^2 A_X = -(\gamma + 3\lambda K_0 K_{0X} - 3\lambda K_0^2 K_1 + \Omega_1)A. \tag{2.30}$$

Using (2.29) and (2.30) in (2.28) we can solve to obtain

$$U_1 = B(X,T)\exp[-H] . \tag{2.31}$$

From equations (2.27) and (2.29), and (2.31) we see that it is not necessary to expand H, the phase, since the order ε and higher order terms can be absorbed into the amplitudes. Thus equations (2.29) and (2.30) become

$$\Omega = \lambda K^3 , \tag{2.32}$$

and

$$A_T + 3\lambda K^2 A_X = -(\gamma + 3\lambda K K_X)A. \tag{2.33}$$

Equation (2.32) is the "dispersion relation" for exponential tails for the linearized KdV equation. Using this relation, equation (2.24) for the conservation of waves becomes

$$K_T + 3\lambda K^2 K_X = -\lambda_X K^3 , \tag{2.34}$$

which must be solved along with equation (2.33). The leading tail solution must then be matched to the core solution. Under physically reasonable conditions, we will show in Chapter III, using the method of characteristics for equation (2.34), that characteristics will cross causing a caustic to form. Then we will consider this caustic formation and its interpretation.

CHAPTER III

ACCELERATING PISTON PROBLEM

3.1 Introduction

In their now classic monograph [3], Courant and Friedrichs briefly discuss a gas dynamics problem in which a compression wave is produced by a uniformly accelerated piston. Figure 3.1 shows the envelope formed by the crossing of straight characteristics which describe this problem. In this chapter we show that the moving core of a slowly varying solitary wave acts like the piston of this gas dynamics problem. In Section 3.6 we show that the constant coefficient problem, equation (2.1) with constant coefficients, has straight characteristics, which for an accelerating wave form the same envelope as the uniformly accelerated piston (Figure 3.1). If on the other hand the wave is decelerating, then a rarefaction wave is produced as with a uniformly decelerated piston (Figure 3.2). In Section 3.2 we show for the more general case of variable coefficients that an envelope may or may not form depending on the medium and on the motion of the wave. The usual case of an accelerating wave with variable coefficients is illustrated in Figure 3.3, where the characteristics are now curved due to the variable coefficients.

We also show in Section 3.2 how the characteristics arise and can lead to a focusing solution. Then, in Section 3.3, we utilize a

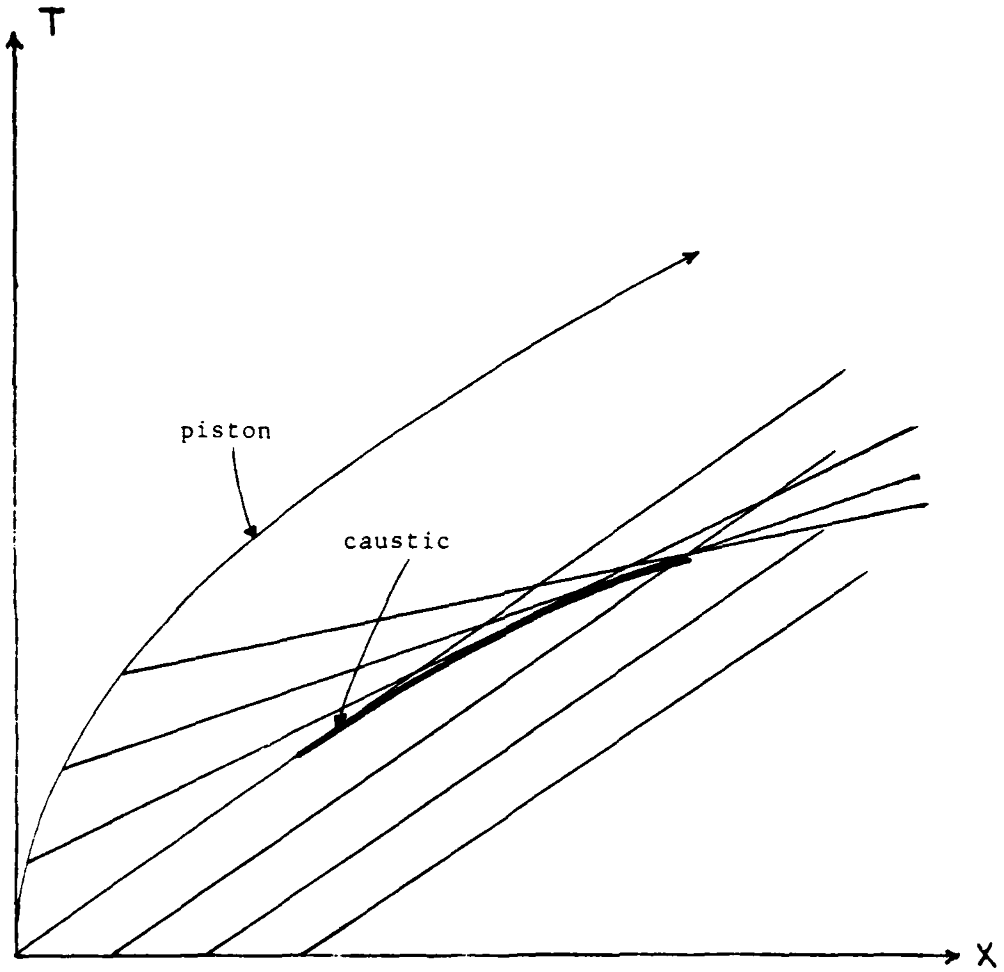

Figure 3.1 Envelope of straight characteristics for a compression wave produced by a uniformly acceleratec piston [3].

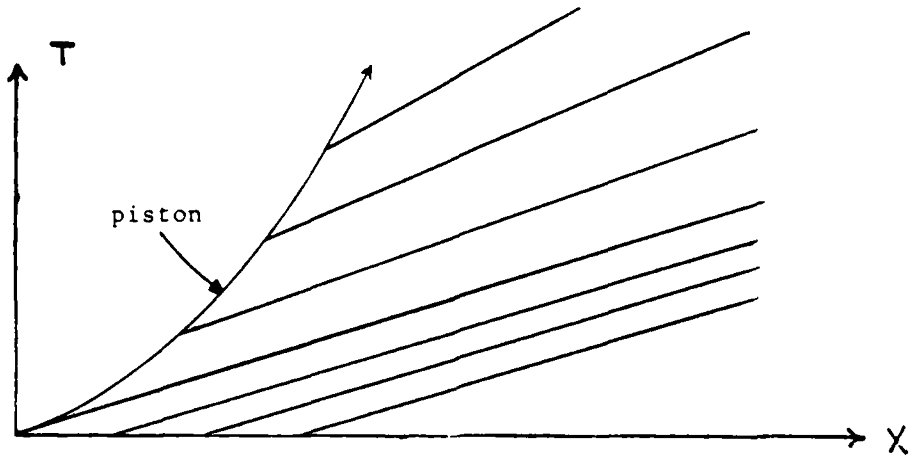

Figure 3.2 Straight characteristics of the rarefaction wave produced by a uniformly decelerated piston [3].

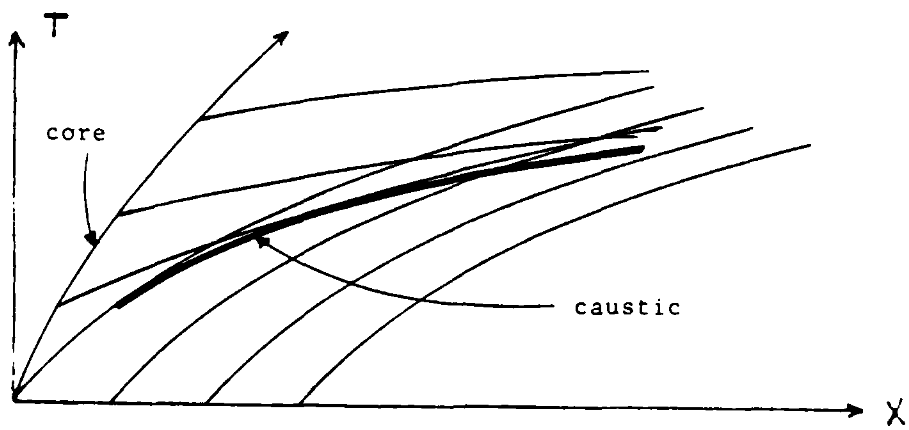

Figure 3.3 Envelope of the curved characteristics for the leading tail of a slowly varying solitary wave propagating on the surface of water of variable depth.

similarity variable to determine equations for the envelope (boundary of the region of multiple characteristics), and to obtain a "fundamental folding equation". Examining the local behavior near first focusing leads us to new scales. We show, in Section 3.4, how these new scales relate the solution to a solution of the diffusion equation. In Section 3.5, we use Laplace's method for the asymptotics of exponential integrals to determine the asymptotic behavior of our solution near first focusing. We find the solution to be a single exponential tail everywhere except in a relatively thin neighborhood of a "shock path", where the solution makes a transition from one exponential tail to another. In this neighborhood the solution is the sum of two tails. The "shock path" is actually the location in space time of a rapid change in wave number or a "wave number shock". This idea of a wave number shock was first discussed for slowly varying solutions of reaction-diffusion equations by Howard and Kopell [10]. Finally, in Section 3.6, we work through an example of our analysis using constant coefficients.

3.2 A Focusing Solution

In the last chapter we determined the leading order solution for both the core

$$V_0 = 3\,\frac{X'_{c_0}}{\hat{\nu}}\ \mathrm{sech}^2\left[\left(\frac{X'_{c_0}}{4\hat{\lambda}}\right)^{1/2}\eta\right], \tag{3.1}$$

and the leading tail (right going with $\lambda > 0$)

$$U_0 = A\exp[-H]\ . \tag{3.2}$$

Using (2.13.b) and the given coefficients ν, λ and γ, we can solve equation (2.20) to completely determine V_0, and thus to leading order

the path of the core, $X_{c_0}(T)$. We now seek the behavior of U_0 by solving the system of pde's given by equations (2.33) and (2.34). First we match U_0 and V_0 using the basic matching principle (see [15,20]). We write V_0 in terms of X and T, U_0 in terms of η and T, and expand for $\varepsilon \to 0$. Thus we have

$$V_0 \sim 12 \frac{X'_{c_0}}{\hat{\nu}} \exp[-(\frac{X'_{c_0}}{\hat{\lambda}})^{1/2} \frac{X-X_c}{\varepsilon}] ,$$

and

$$U_0 \sim \hat{A} \exp[-\hat{K} \frac{X-X_c}{\varepsilon}] ,$$

where " ^ " again means evaluation at $X = X_{c_0}$. Now the matching of U_0 and V_0 yeilds the matching conditions

$$\hat{A} = 12 \frac{X'_{c_0}}{\hat{\nu}} , \tag{3.3}$$

and

$$\hat{K} = \left(\frac{X'_{c_0}}{\hat{\lambda}}\right)^{1/2} . \tag{3.4}$$

Next we use the method of characteristics to solve the leading tail equations (2.33) and (2.34):

$$A_T + 3\lambda K^2 A_X = (\gamma + 3\lambda K K_X)A ,$$

and

$$K_T + 3\lambda K^2 K_X = -\lambda_X K^3 .$$

Taking

$$\frac{dX}{dT} = 3\lambda K^2, \tag{3.5}$$

we see, using equation (3.4), that near the core the characteristic velocity for the leading tail is three times the core velocity. This means that disturbances, in the leading tail near the core, will move away from the core into the tail. Thus, the moving core acts as a space-like boundary for the leading tail. The problem of solving the leading tail equations subject to a specified core (amplitude and phase) is well-posed; there is a unique leading tail matching to the core. On the characteristics our system of pde's reduces to a system of ode's

$$\frac{dA}{dT} = -(\gamma + 3\lambda K K_X)A \ , \tag{3.6}$$

and

$$\frac{dK}{dT} = -\lambda_X K^3 \ . \tag{3.7}$$

Equations (3.5) and (3.7) imply that the characteristics are given by

$$X = F(T;\xi) \ , \tag{3.8}$$

where ξ is the parameterization of time along the path of the leading order core (at $T = \xi$, $X = X_{c_0}(\xi)$):

$$F(\xi;\xi) = X_{c_0}(\xi), \qquad \xi \geq 0 \ . \tag{3.9}$$

We call the crossing of characteristics focusing, and the locus of all such focusing points is a caustic. The caustic is obtained by simultaneously solving (3.8) and

$$F\,(T_f;\xi) = 0 \ . \tag{3.10}$$

The times, $T = T_f$, for which focusing occurs, due to the moving core, are obtained by solving equation (3.10) for T_f:

$$T_f = T_f(\xi) \ . \tag{3.11}$$

Focusing occurs ahead of the core if

$$T_f(\xi) > \xi.$$

The corresponding positions at which focusing occurs are then determined from equation (3.8):

$$X_f = F(T_f;\xi) \ . \tag{3.12}$$

Equations (3.11) and (3.12) are the parametric representation of the caustic due to the moving core.

Thus far we have considered only those characteristics eminating from the path of the core (see Figure 3.4). We now consider the characteristics due to initial conditions which eminate from the positive X-axis (see Figure 3.5). Equations (3.5) and (3.7) imply that these characteristics are given by

$$X = G(T;\zeta) \ , \tag{3.13}$$

where ζ is the parametrization of initial position (at $T=0$, $X=\zeta$):

$$G(0;\zeta) = \zeta \ , \ \zeta > 0.$$

Analyzing these characteristics as above we see that they too can focus. We do not pursue the case where neither (3.8) nor (3.13) leads to focusing since there is then no difficulty with our asymptotic expansions. We proceed under the assumption that there is focusing due to the movement of the core only, and will consider the other interesting case of focusing due to initial conditions in Chapter IV (see also [8]).

Returning to the characteristics, (3.8), we determine the time of first focusing, $T=T_1$, by minimizing (3.11) for $\xi \geq 0$. There are two possibilities. First we can have

$$T_1 = T_f(0), \tag{3.14}$$

with

$$T_f'(0) \geq 0 \ . \tag{3.15}$$

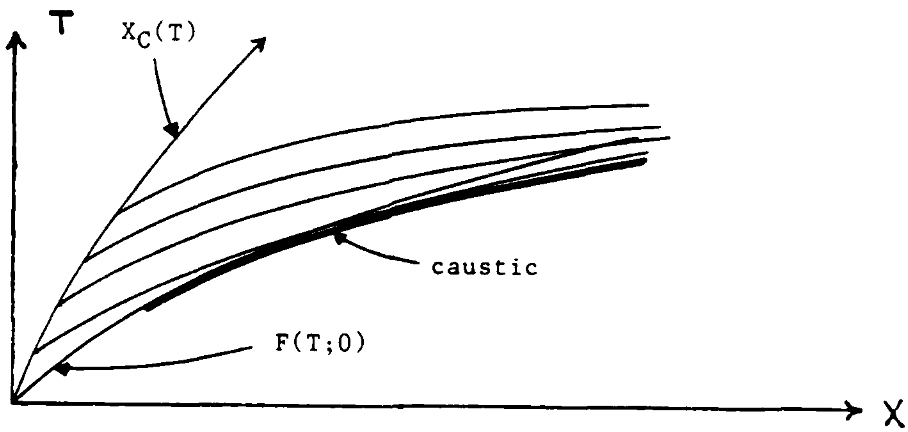

Figure 3.4 Characteristics eminating from the path of the core (due to the movement of the core).

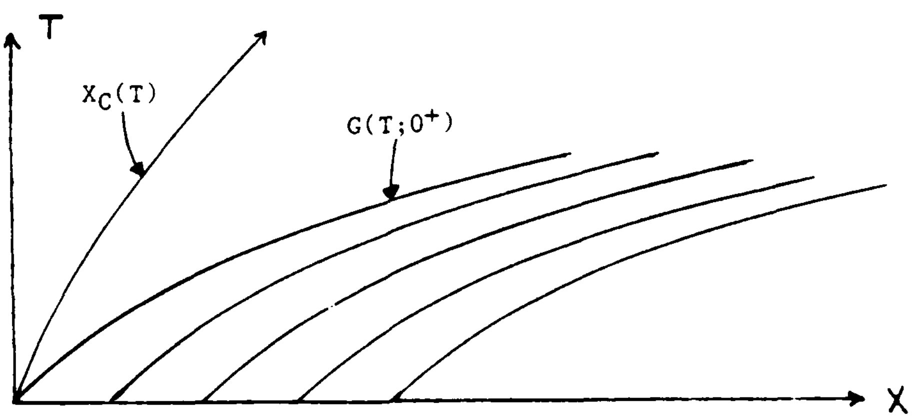

Figure 3.5 Characteristics eminating from the X-axis (due to initial conditions).

In this case first focusing occurs along the characteristic, $\xi=0$, eminating from the origin; focusing due to later movement of the core happens at a later time (thus giving rise to a penumbral caustic, see Fig. 3.4 and [20]). The other possibility is

$$T_1 = T_f(\xi_1) \ , \tag{3.16}$$

with

$$T_f'(\xi_1) = 0 \text{ (usually with } T_f''(\xi_1) > 0) \ . \tag{3.17}$$

In this case first focusing occurs along the characteristic, $\xi = \xi_1 > 0$; focusing due to earlier and later movement of the core occurs later (thus giving rise to a cusped caustic). We consider the first possibility, equations (3.14) and (3.15), in the remainder of this chapter. The second possibility, equations (3.16) and (3.17), will be covered in Chapter IV along with focusing due to initial conditions.

3.3 First Focusing from the Initial Core Position (Local Behavior)

We now consider the case where first focusing occurs due to the immediate movement of the core from its initial position, that is equations (3.14) and (3.15) apply (of course the time to first focus is finite, $T_f(0)>0$). We approximate the equation for the characteristics near first focusing by Taylor expanding (3.8) around $\xi=0$ and $T=T_1=T_f(0)$. In this way we obtain the "fundamental folding equation":

$$X-X_1-F_T(T_1;0)(T-T_1) - \frac{1}{2}F_{TT}(T_1;0)(T-T_1)^2 =$$

$$F_{T\xi}(T_1;0)(T-T_1)\xi + \frac{1}{2}F_{\xi\xi}(T_1;0)\xi^2, \tag{3.18}$$

a cut off quadratic in ξ. From equations (3.10) and (3.11) we see that

$$\frac{d}{d\xi} F_\xi(T_f;\xi) = F_{T\xi}(T_f;\xi)T_f'(\xi) + F_{\xi\xi}(T_f;\xi) = 0 . \tag{3.19}$$

Thus if $T_f'(0) = 0$, then $F_{\xi\xi}(T_1;0) = 0$ and we would need to include cubic terms in the approximation (3.18). The analysis for the cubic will be done in Chapter IV. We now proceed under the assumption that $T_f'(0) > 0$. Equation (3.18) is valid for $\xi \geq 0$, since the parameterization for the position of the core begins at $\xi = 0$. We notice from equation (3.18) that the change of variable

$$\xi = (T-T_1)f(S), \tag{3.20}$$

shows that $f(S)$ is a function of the similarity variable S, where

$$S \equiv [X-X_1 - C_g(T-T_1) - C(T-T_1)^2]/(T-T_1)^2, \tag{3.21}$$

with the constants C_g and C defined by

$$C_g \equiv F_T(T_1;0) ,$$

and

$$C \equiv \frac{1}{2}F_{TT}(T_1;0) .$$

Thus the approximation for the characteristics, (3.18), becomes

$$af^2(S) + bf(S) = S , \tag{3.22}$$

where the constants a and b are defined by

$$a \equiv \frac{1}{2} F_{\xi\xi}(T_1;0),$$

and

$$b \equiv F_{T\xi}(T_1;0).$$

From equation (3.19) we then have

$$a = -\frac{1}{2} T_f'(0)b. \tag{3.23}$$

Now, for $T<T_1$, $X-X_1 < C_g(T-T_1) + C(T-T_1)^2$, and $\xi > 0$, we see from (3.21)

that $S < 0$. From (3.22) we have

$$f(S) = [-b \pm \sqrt{b^2 + 4aS}]/2a \;, \tag{3.24}$$

and (3.20) implies $f(S) < 0$. In this region of space time there is no crossing of characteristics, so there must be a single characteristic for each S. From (3.24) we see that in order for f(S) to have a single negative value, and thus from (3.20) for ξ to have a single positive value, we must have $aS > 0$ or $a < 0$. Equation (3.23) then yields $b > 0$. Examining equations (3.20) and (3.24) for $T > T_1$ shows that these signs for a and b yield the required two positive values of f(S) in the appropriate region of crossing characteristics. Physically, $b > 0$ means the velocity of the leading tail is increasing, with respect to the position of the core, at first focusing ($\xi = 0$ and $T = T_1$).

Since T_1 is the time of first focusing, all focusing occurs for times $T \geq T_1$, and thus the envelope of crossing characteristics is, given by equation (3.24), the boundary of the region where f(S) has two positive values:

$$S = 0, \; T \geq T_1 \;,$$

and

$$S = -\frac{b^2}{4a} \;, \quad T \geq T_1 \;.$$

The neighborhood of first focusing can be divided into three regions according to the number of characteristics passing through each point (see Figures 3.3 and 3.6). Region I is given by

$$X < X_1 + C_g(T-T_1) + C(T-T_1)^2 \;,$$

and has a single characteristic, eminating from the moving core, passing through each point, i.e. f(S) has one value satisfying both (3.20) and

(3.22). Region II is given by

$$X_1 + C_g(T-T_1) + C(T-T_1)^2 < X < X_1 + C_g(T-T_1) + (C - \frac{b^2}{4a})(T-T_1)^2, \; T>T_1.$$

This region has two characteristics, eminating from the moving core, passing through each point, i.e. $f(S)$ has two values satisfying both (3.20) and (3.22), and a third characteristic, eminating from the initial conditions, also passes through each point. Region III is given by

$$X > X_1 + C_g(T-T_1) + C(T-T_1)^2, \quad T < T_1 ,$$

or

$$X > X_1 + C_g(T-T_1) + (C - \frac{b^2}{4a})(T-T_1)^2, \quad T > T_1.$$

This region has a single characteristic, eminating from the initial conditions, passing through each point and no value of $f(S)$ satisfying both (3.20) and (3.22). Figure 3.6 shows these regions, and Figure 3.7 gives the evolution of the number of characteristic values at each X due to the moving core.

We now seek the behavior of K and A near first focusing, $X=X_1$ and $T=T_1$. From equation (3.8) we have

$$K(X,T) = K[F(T;\xi),T] = \bar{K}(T;\xi) ,$$

and expanding in a Taylor series around $T=T_1$ and $\xi = 0$

$$K = \bar{K}(T_1;0) + \bar{K}_T(T_1;0)(T-T_1) + \bar{K}_\xi(T_1;0)\xi + \ldots \quad .$$

Substituting (3.20) into this expression yields

$$K = \bar{K}(T_1;0) + [\bar{K}_T(T_1;0) + \bar{K}_\xi(T_1;0)f(S)](T-T_1) + \ldots, \tag{3.25}$$

and thus as $T \to T_1$ (keeping S fixed so that $\xi \to 0$)

$$K_X \sim \bar{K}_\xi(T_1;0)f'(S)/(T-T_1) \; . \tag{3.26}$$

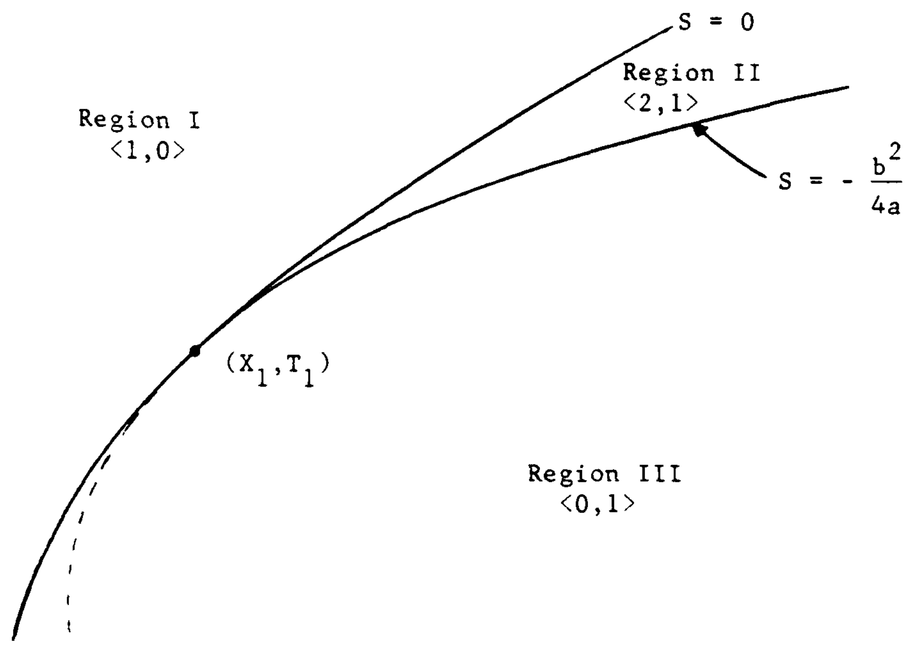

Figure 3.6 Number of characteristics passing through each point in regions of the neighborhood of first focusing due to: <moving core, initial conditions>.

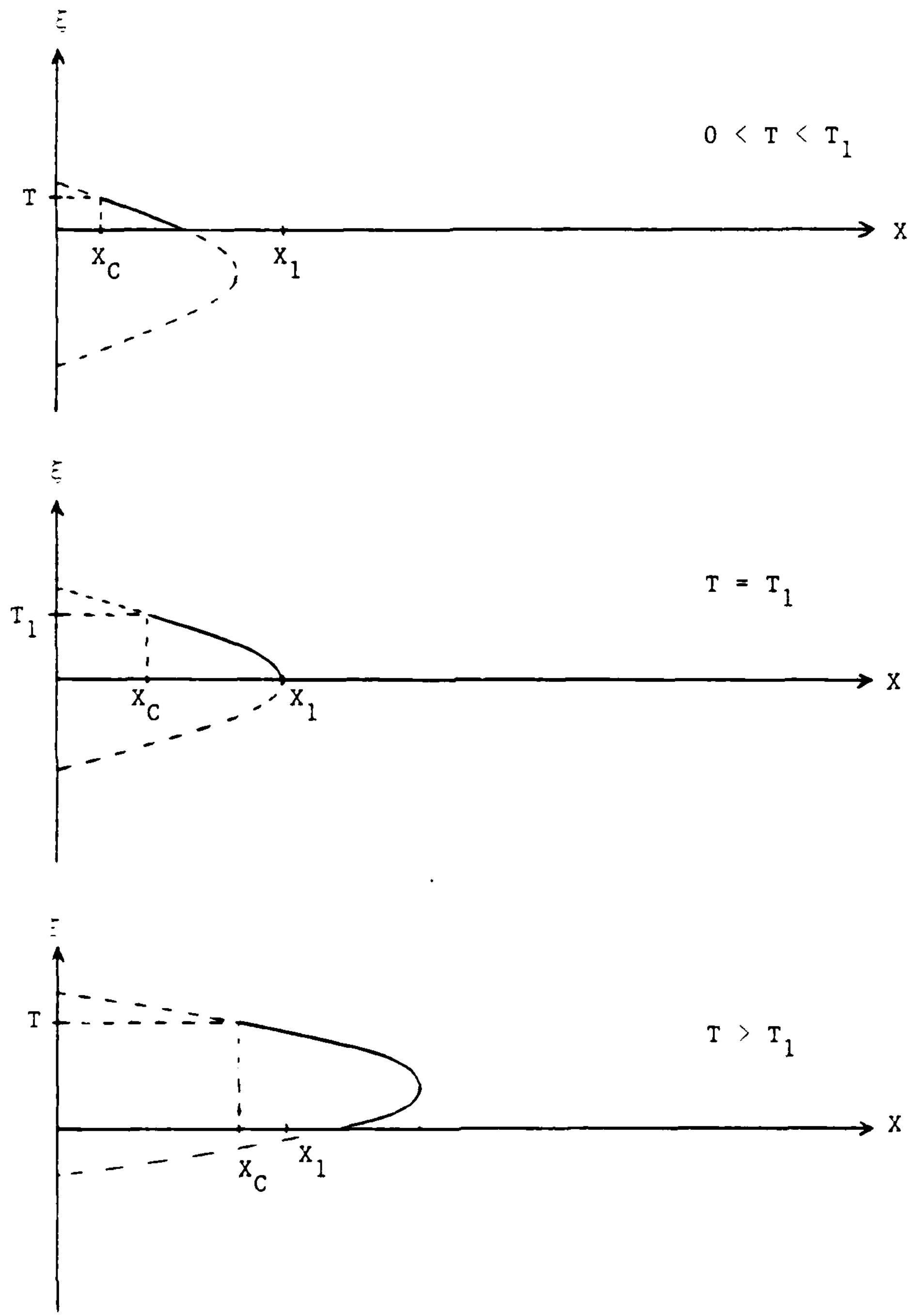

Figure 3.7 Evaluation of the number of characteristics at each X due to the moving core.

From equations (3.5) and (3.18) we have

$$3\lambda K^2 \sim C_g + b\xi ,$$

and thus

$$3(\lambda_X K^2 + 2\lambda K K_X) \sim b\xi_X . \qquad (3.27)$$

For well behaved λ, (3.25), (3.26) and (3.27) yield

$$6\lambda K K_X \sim b\xi_X ,$$

and from (3.18)

$$6\lambda K K_X \sim \frac{b}{b(T-T_1) + 2a\xi} . \qquad (3.28)$$

Thus, equations (3.6) and (3.28) yield

$$\frac{dA}{dT} \sim -\frac{1}{2}\,\frac{bA}{b(T-T_1) + 2a\xi} ,$$

or

$$A \sim \bar{A}(\xi)\,|b(T-T_1) + 2a\xi|^{-1/2} ,$$

where $\bar{A}(\xi)$ is determined by the matching condition (3.3). Using (3.20) we now have

$$A \sim \bar{A}(\xi)\,|[b + 2af(S)](T-T_1)|^{-1/2} . \qquad (3.29)$$

The singular behavior of A at first focusing suggests the use of different scales, which we introduce in the following section.

3.4 A Relation to the Diffusion Equation

From equation (2.25) we obtained the leading order equation (2.26) with solution

$$U_0 = A(X,T)\,\exp[-H] . \qquad (3.30)$$

Near first focusing A is singular as given by (3.29). We will show higher order terms in the expansion are more singular. At the next order we obtained the solution

$$U_1 = B(X,T)\exp[-H] . \tag{3.31}$$

The next order equation is very important:

$$- \Omega U_{2H} + \lambda K^3 U_{2HHH} = -\gamma U_1 - U_{1T} - \lambda K_{XX} U_{0H}$$

$$- 3\lambda K K_X U_{1HH} - 3\lambda K_X U_{0XH} - 3\lambda K^2 U_{1XHH} - 3\lambda K U_{0XXH} \tag{3.32}$$

We see that the right hand side produces secular terms for U_2 unless

$$B_T + 3\lambda K^2 B_X = -(\gamma + 3\lambda K K_X)B + \lambda K_{XX} A$$

$$+ 3\lambda K_X A_X + 3\lambda K A_{XX} . \tag{3.33}$$

The asymptotic behavior of A, (3.29), leads us to assume

$$B \sim |T-T_1|^P \, f(S). \tag{3.34}$$

Substituting (3.25), (3.29) and (3.34) into (3.33) yields P = - 7/2. Continuing this process for higher orders shows that the expansion for U breaks down when

$$T-T_1 = O(\varepsilon^{1/3}) ,$$

which suggests the scaling

$$\tau = \varepsilon^{-1/3}(T-T_1) = \varepsilon^{2/3}(t-t_1) . \tag{3.35}$$

In order to keep the similarity variable, S, order one, we must take

$$Z = \varepsilon^{-2/3}[X-X_1-C_g(T-T_1)-C(T-T_1)^2]$$

$$= \varepsilon^{1/3}[x-x_1-C_g(t-t_1)-\varepsilon C(t-t_1)^2] . \tag{3.36}$$

Equations (3.35) and (3.36) are the appropriate independent variables in a neighborhood near first focusing. We note that the scales of these variables are intermediate to the other two scalings, large compared to the scales of the solitary wave and small compared to the scales of the slowly varying media. Using equations (2.22), (2.23) and (2.32), near first focusing we have

$$H = \frac{1}{\varepsilon}[\theta(X_1,T_1) + K(X_1,T_1)(X-X_1) - \lambda(X_1,T_1)K^3(X_1,T_1)(T-T_1) + \ldots]$$

$$= \frac{1}{\varepsilon}[\theta^{(1)}-K^{(1)}X_1 + \lambda^{(1)}K^{(1)^3}T_1] + K^{(1)}[x-\lambda^{(1)}K^{(1)^2}t] + \ldots, \tag{3.37}$$

where the superscript "(1)" means evaluated at $X=X_1$ and $T=T_1$. Since, from (3.29), the amplitude is singular at first focusing, equations (3.30) and (3.37) motivate us, using the intermediate scales (3.35) and (3.36), to seek a solution of the form

$$\bar{u} = \phi(Z,\tau)\exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)] , \tag{3.38}$$

where the large constant in (3.37) will be part of $\phi(Z,\tau)$. This is a different multiple scale approach than the one we began with in Section 2.2 (equation (2.3)). Thus we have

$$u(x,t) = \bar{u}(x,t,Z,\tau) , \tag{3.39}$$

$$u_t = [\lambda^{(1)}K^{(1)^3}\phi - \varepsilon^{1/3}C_g\phi_Z + \varepsilon^{2/3}\phi_\tau - 2\varepsilon^{4/3}C(t-t_1)\phi_Z]$$

$$\exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2} t)] ,$$

and

$$U_{xxx} = [-K^{(1)^3}\phi + 3\varepsilon^{1/3}K^{(1)^2}\phi_Z - 3\varepsilon^{2/3}K^{(1)}\phi_{ZZ} + \varepsilon\phi_{ZZZ}]$$

$$\exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)] .$$

Substituting these values into equation (2.21) we have

$$\phi_\tau = 3\lambda^{(1)}K^{(1)}\phi_{ZZ} + \varepsilon^{1/3}\{[\lambda_X^{(1)}(x-x_1) + \lambda_T^{(1)}(t-t_1)]K^{(1)^3}\phi$$

$$- \lambda^{(1)}\phi_{ZZZ} - \gamma^{(1)}\phi\} + \text{H.O.T.}$$

Near first focusing the leading order term for satisfies the diffusion equation

$$\phi_{0_\tau} = 3\lambda^{(1)}K^{(1)}\phi_{0_{ZZ}} . \qquad (3.40)$$

We showed in Chapter II that for the right going wave, which we have considered here, $\lambda > 0$, and hence (3.40) is the well-posed diffusion equation. It is extremely interesting to note that even if $\gamma \equiv 0$ we still easily obtain the diffusion equation from the KdV equation even though the KdV equation is dispersive with no dissipation! If we momentarily consider the crossing of characteristics for the trailing tail of a right going wave, then the analysis is nearly identical to the above. However, equation (3.38) becomes

$$\bar{u} = \phi \exp[K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)],$$

and leads to the backwards diffusion equation, which is ill-posed. In this case the solution is grossly unstable, and therefore the assumption of a slowly varying exponentially decaying tail is <u>not</u> valid for the trailing tail of a right going wave. In fact, the trailing tail is known to not be exponentially decaying. Ko and Kuehl [16, 17], Kaup and Newell [14], Karpman and Maslov [13], Knickerbocker and Newell [19], Grimshaw [5], Kodama and Ablowitz [17], and Smyth [28] show that a "shelf" exists there and thus the prediction of our analysis is consistent.

We now return to the discussion of the leading tail of a right going wave. We wish to solve equation (3.40) subject to matching the leading order singular solution (3.30). Details of the matching will be presented in Section 3.5 where we will show that the desired matching solution of (3.40) is a superposition of exponentials:

$$\phi_0(Z,\tau) = \int_{-\infty}^{\infty} D(\alpha)\exp[-\alpha Z + 3\lambda^{(1)}K^{(1)}\alpha^2\tau]d\alpha, \tag{3.41}$$

where

$$D(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \bar{D}\exp[12a(\frac{\lambda^{(1)}K^{(1)}}{b})^2\alpha^3], & \alpha > 0, \end{cases}$$

and $\bar{D}$ is determined by matching. Equation (3.41) obviously solves the diffusion equation. The substitution

$$\alpha = \frac{\varepsilon^{-1/3}b}{6\lambda^{(1)}K^{(1)}}\xi,$$

into (3.41) yields

$$\phi_0 = \frac{\varepsilon^{-1/3}b\bar{D}}{6\lambda^{(1)}K^{(1)}}\int_0^{\infty} \exp[\frac{b}{6\varepsilon\lambda^{(1)}K^{(1)}}(-\varepsilon^{2/3}Z\xi + \frac{1}{2}\varepsilon^{1/3}b\tau\xi^2 + \frac{1}{3}a\xi^3)d\xi \tag{3.42}$$

a convergent integral since $a < 0$. It is an exponential integral with a cubic phase, an incomplete Airy integral of the type which would arise in the analysis of the formation of a shock wave for a piston (Lighthill [22], although no details are presented).

Using Laplace's method for the asymptotics of exponential integrals we will show in Section 3.5 that (3.42) is the desired matching solution. In addition we will show that the critical points of the exponential phase satisfy the fundamental folding incomplete

quadratic (3.18).

We point out here that if we solve equation (3.40) as an initial value problem (which, it is not), with the initial condition given by equation (3.41) with $\tau = 0$, then the fundamental solution of the diffusion equation yields the solution we have obtained by matching. For details of this relationship see the Appendix.

3.5 Asymptotics and the "Shock Path"

We now show that the leading order solution near first focusing

$$\bar{u}_0 = \phi_0 \exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)] \ , \tag{3.43}$$

matches to the leading order singular solution

$$U_0 = A \exp[-H] \ . \tag{3.44}$$

The basic matching principle is to express each solution in terms of the other variables, expand for small ε, and equate the resulting expansions to obtain the matching conditions (see for example [15, 23]). To change variables we use the scaling equations:

$$\tau = \varepsilon^{-1/3}(T-T_1) = \varepsilon^{2/3}(t-t_1) \quad ,$$

and

$$Z = \varepsilon^{-2/3}[X-X_1-C_g(T-T_1)-C(T-T_1)^2]$$

$$= \varepsilon^{1/3}[x-x_1-C_g(t-t_1)- C(t-t_1)^2] \ ,$$

from Section 3.4. Solving for T and X we have

$$T = T_1 + \varepsilon^{1/3} \quad ,$$

and

$$X = X_1 + \varepsilon^{1/3}C_g\tau + \varepsilon^{2/3}(Z + C\tau^2) \ .$$

Thus, for small ε we have

$$H \sim \frac{1}{\varepsilon}[\vartheta^{(1)} + K^{(1)}(\varepsilon^{2/3}Z + \varepsilon^{1/3}C_g\tau + \varepsilon^{2/3}C\tau^2)$$

$$- \lambda^{(1)}K^{(1)^3}(\varepsilon^{1/3}\tau)] \ , \tag{3.45}$$

and from (3.29)

$$A \sim \bar{A}(\xi)|[b + 2af(S)]\varepsilon^{1/3}\tau|^{-1/2} \quad . \tag{3.46}$$

From (3.42) we have

$$\phi_0 = \varepsilon^{-1/3}d\bar{D}\int_0^\infty \exp[\frac{d}{\varepsilon}\,h(\xi)]d\xi \ , \tag{3.47}$$

where

$$d \equiv \frac{b}{6\lambda^{(1)}K^{(1)}} \quad ,$$

and

$$h(\xi) \equiv \frac{1}{3}\,a\xi^3 + \frac{1}{2}b(T-T_1)\xi^2 - [X-X_1-C_g(T-T_1)-C(T-T_1)^2]\xi \quad .$$

We now use Laplace's method for the asymptotics of exponential integrals on (3.47). The critical points of the cubic phase are determined by solving

$$h'(\xi_c) = a\xi_c^2 + b(T-T_1)\xi_c - [X-X_1-C_g(T-T_1)-C(T-T_1)^2] = 0 \ , \tag{3.48}$$

for ξ_c. Equation (3.48) is the fundamental folding incomplete quadratic equation (3.18)! Using (3.21) and solving (3.48) we have

$$\xi_c = -\frac{b}{2a}(T-T_1)[1 \pm \sqrt{1 + \frac{4a}{b^2}S}\,], \quad S \leq -\frac{b^2}{4a} \quad . \tag{3.49}$$

Since $h(\xi)$ is a cubic with a negative leading coefficient, its local maximum must be to the right of its local minimum (when they exist for

$\xi \geq 0)$, or

$$\xi_{max} = -\frac{b}{2a}(T-T_1) \begin{cases} 1 - \sqrt{1 + \frac{4a}{b^2} S}, & T < T_1 \text{ and } S \leq 0 \\ 1 - \sqrt{1 + \frac{4a}{b^2} S}, & T > T_1 \text{ and } S \leq \frac{-b^2}{4a} . \end{cases}$$

We now have

$$h(\xi_{max}) = \frac{b^3}{24a^2}(T-T_1)^3 \begin{cases} -1+3(1 + \frac{4a}{b^2} S) - 2(1 + \frac{4a}{b^2} S)^{3/2}, & T<T_1 \text{ and } S \leq 0 \\ -1+3(1 + \frac{4a}{b^2} S) + 2(1 + \frac{4a}{b^2} S)^{3/2}, & T>T_1 \text{ and } S \leq \frac{-b^2}{4a} \end{cases}$$

$$h'(\xi_{max}) = 0 ,$$

$$h''(\xi_{max}) = b(T-T_1) \begin{cases} \sqrt{1 + \frac{4a}{b^2} S} , & T < T_1 \text{ and } S \quad 0 \\ -\sqrt{1 + \frac{4a}{b^2} S} , & T > T_1 , \text{ and } S \quad -\frac{b^2}{4a}. \end{cases}$$

In Figure 3.8 we have extended the domain of h to illustrate its evolution throughout the neighborhood of first focusing. The location of ξ_{max} and the magnitude of $h(\xi_{max})$ are critical in the determination of the asymptotics of ϕ_0 using Laplace's method. We must keep in mind that $\xi=0$ is a cut-off value for the integral in ϕ_0. This cut-off gives rise to a penumbral caustic in contrast to a more common cusped caustic (for further discussion of the classification of caustics we refer the reader to [10]). In order for the asymptotics to be affected by ξ_{max} we must have $\xi_{max} > 0$ and $h(\xi_{max}) \geq h(0) = 0$, otherwise the asymptotics will be completely determined by the endpoint of the integral, $\xi = 0$. Laplace's method, for fixed $S > 0$ and $T < T_1$ in (3.47), yields

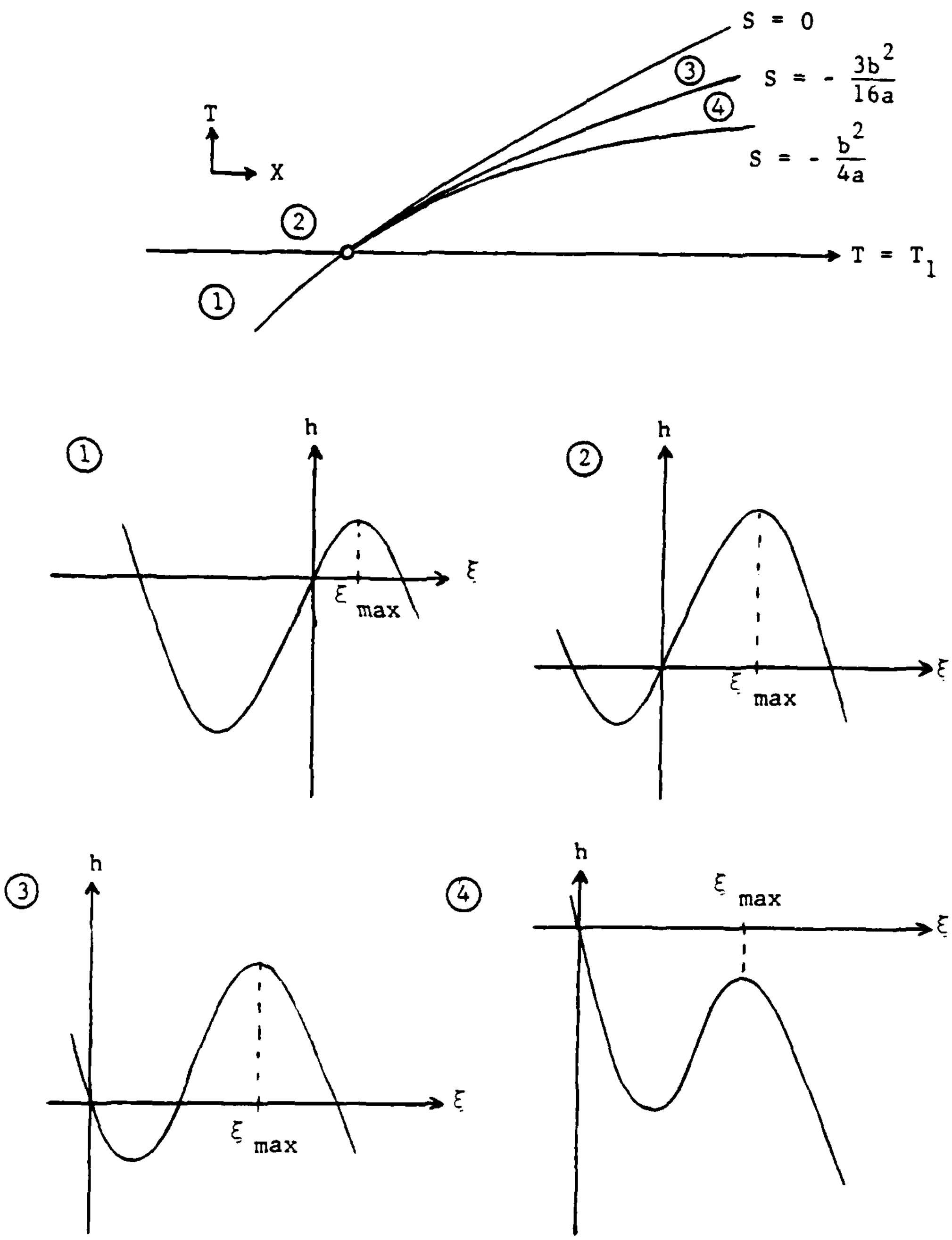

Figure 3.8 Evolution of exponential phase, $h(\xi)$, near first first focusing with domain of h extended to negative ξ for illustration only. Laplace's method for equation (3.42) uses the maximum of $h(\xi)$ for $\xi > 0$.

$$\phi_0 \sim \varepsilon^{-1/3} d\bar{D}\int_{-\infty}^{\infty} \exp\{\frac{d}{\varepsilon}[h(\xi_{max}) + \frac{1}{2}h''(\xi_{max})(\xi-\xi_{max})^2]\}d\xi$$

$$\sim \bar{D}[\frac{-2\varepsilon^{1/3} d}{h''(\xi_{max})}]^{1/2} \exp[\frac{d}{\varepsilon}h(\xi_{max})] . \tag{3.50}$$

Now using (3.45), (3.46) and (3.50) in equations (3.43) and (3.44) yields the matching condition

$$\bar{D} = \bar{A}(0)[2\varepsilon^{1/3} d\pi]^{-1/2} \exp\{\frac{1}{\varepsilon}[\theta^{(1)} - K^{(1)}(X_1 - \lambda^{(1)} K^{(1)^2} T_1)]\} \tag{3.51}$$

Having thus determined $\bar{D}$ we can now obtain the asymptotics of ϕ_0 for $T > T_1$ and fixed $S \leq -b^2/4a$. For fixed $S < -3b_2/(16a)$, $\xi = \xi_{max}$ dominates the asymptotics (see Figure 3.8) as before and Laplace's method again yields (3.50). However, for S fixed and $-3b^2/(16a) < S \leq -b^2/(4a)$, the endpoint $\xi = 0$, dominates the asymptotics (see Figure 3.8) and thus using Laplace's method for an endpoint contribution we find from (3.47) that

$$\phi_0 \sim \frac{\bar{D}}{Z} . \tag{3.52}$$

The most interesting case is when both $\xi = 0$ and $\xi = \xi_{max} > 0$ contribute to the asymptotics. This occurs when $h(\xi_{max}) = h(0) = 0$ or $S = -3b^2/(16a)$. Laplace's method then yields

$$\phi_0 \sim \frac{\bar{D}}{Z} + \bar{D}[\frac{-2\varepsilon^{1/3} d}{h''(\xi_{max})}]^{1/2} . \tag{3.53}$$

This clearly demonstrates that, near first focusing, there is a transition region where the asymptotic solution is determined both by the moving core, $\xi = \xi_{max} > 0$, and by the initial conditions, $\xi = 0$. This in fact is the initiation of a "shock path". The solution remains continuous, but there is an abrupt change in wave number or a wave

number shock. Howard and Kopell [10] have discussed a similar concept for slowly varying solutions of reaction-diffusion equations. The shock path near first focusing (and $T>T_1$) is determined from

$$S_s = [X_s - X_1 - C_g(T-T_1) - C(T-T_1)^2]/(T-T_1)^2 = -\frac{3b^2}{16a},$$

the points where the asymptotic are determined from both the moving core and the initial conditions. The shock path near first focusing is thus

$$X_s = X_1 + C_g(T-T_1) + (C - \frac{3b^2}{16a})(T-T_1)^2, \tag{3.54}$$

with shock speed

$$\frac{dX_s}{dT} = C_g + 2(C - \frac{3b^2}{16a})(T-T_1). \tag{3.55}$$

As we entered the region near first focusing ($T < T_1$), we matched the leading tail solution to the solution near first focusing. This matching determined the solution near first focusing. We must match this solution to the solution after first focusing ($T > T_1$). This matching is identical to the matching prior to first focusing. Thus we see that, after first focusing, the moving core contributes a single exponentially decaying tail to the leading tail solution. Since in this chapter we are considering the case with no crossing of characteristics due to the initial conditions, it is easy to show that these initial conditions also contribute a single exponentially decaying tail to the leading tail solution. Therefore, for each space-time point ahead of the core where a single characteristic passes, the leading order solution is a single exponentially decaying tail. However, inside the envelope of crossing characteristics (see Figure 3.3) superposition yields the leading order solution:

$$u(x,t) \sim A_\ell \exp[-H_\ell] + A_r \exp[-H_r] \ , \tag{3.56}$$

where the subscripts "ℓ" and "r" denote the solutions due to the moving core and initial conditions respectively. Equation (3.56) is a linear multi-phase slowly varying tail (see [9, 24]). The asymptotic solution for the leading tail is a single tail everywhere except at or near where the phases in (3.56) are equal, or

$$H_\ell = H_r \quad . \tag{3.57}$$

Applying the chain rule to $\theta = \varepsilon H$ we have

$$\frac{d\theta}{dT} = K\frac{dX}{dT} - \Omega \ . \tag{3.58}$$

Taking $X = X_s(T)$ to be the points where the two phases are equal, we have

$$K_\ell \frac{dX_S}{dT} - \Omega_\ell = K_r \frac{dX_S}{dT} - \Omega_r \ ,$$

or

$$\frac{dX_S}{dT} = \frac{\Omega_\ell - \Omega_r}{K_\ell - K_r} = \frac{[\Omega]}{[K]} \quad . \tag{3.59}$$

This is the well known Rankine-Hugoniot relation for conservation of waves, equation (2.24). Using (2.32), equation (3.59) becomes

$$\frac{dX_S}{dT} = \lambda(K_\ell^2 + K_\ell K_r + K_r^2) \ , \tag{3.60}$$

and as $T \to T_1$ we see that (3.55) and (3.59) do match since

$$C_g = 3\lambda^{(1)} K^{(1)^2} .$$

Figure 3.9 illustrates the asymptotic solution near the wave number stock.

3.6 Constant Coefficients

We now illustrate our analysis for the special case where ν, λ and γ are constant in equation (2.1). Thus (2.20) becomes

$$\frac{dw_0}{dT} = -\frac{4}{3}\gamma w_0 ,$$

or

$$w_0(T) = w_0(0)\exp[-\frac{4}{3}\gamma T] , \tag{3.61}$$

which agrees with Karpman and Maslow [13], and Ablowitz and Segur [1]. The leading order core solution, (2.17), is now completely determined, and from (2.13.b) and (3.61) the center of the core is located at

$$X_{c_0}(T) = \frac{3}{4\gamma}[w_0(0) - w_0(T)] . \tag{3.62}$$

We assume $\lambda > 0$ and thus from (2.17) we must have $w_0(T) > 0$, which means the core moves to the right.

Now turning to the leading tail we see that equation (3.7) becomes

$$\frac{dK}{dT} = 0 , \tag{3.63}$$

on the characteristics, (3.8), or

$$K(X,T) = K[X_{c_0}(\xi),\xi] .$$

From the matching condition, (3.4), to leading order we have

$$K(X,T) = [\frac{w_0(0)}{\lambda}]^{1/2} \exp[-\frac{2}{3}\gamma\xi] ,$$

or

$$\lambda K^2 = w_0(\xi) . \tag{3.64}$$

The characteristics, (3.8), are thus

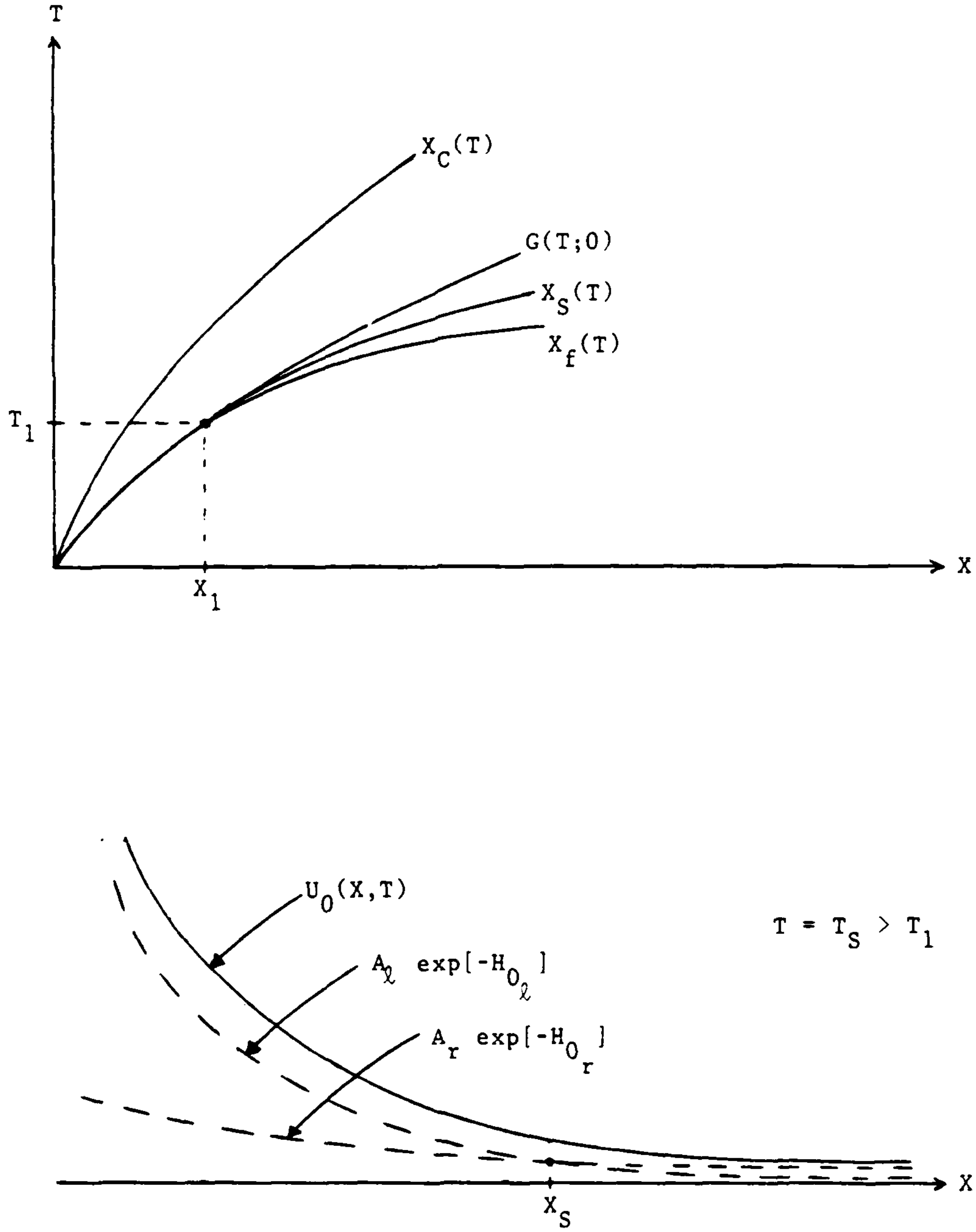

Figure 3.9 "Shock path", $X_s(T)$, of the wave number shock and asymptotic solution as a superposition of left and right tails due to the moving core and initial conditions.

$$X = 3w_0(\xi)(T-\xi) + X_{c_0}(\xi) . \tag{3.65}$$

For focusing we must have

$$3w_0'(\xi)(T_f-\xi) - 3w_0(\xi) + X_{c_0}'(\xi) = 0 ,$$

or

$$T_f(\xi) = \xi - \frac{1}{2\gamma} \quad (\text{thus } T_f'(\xi) = 1 > 0) . \tag{3.66}$$

We see from this relation that we must have $\gamma < 0$ for focusing ahead of the core, thus the core must accelerate! First focusing occurs at

$$T_1 = -\frac{1}{2\gamma} ,$$

and

$$X_1 = -\frac{3w_0(0)}{2\gamma} .$$

From (3.64) we have

$$2\lambda KK_X = w_0'(\xi)\xi_X ,$$

and (3.65) yields

$$\xi_X = \frac{1}{3w_0'(\xi)(T-\xi)-2w_0(\xi)} ,$$

so that

$$3\lambda KK_X = \frac{1}{2}\, \frac{3w_0'(\xi)}{3w_0'(\xi)(T-\xi)-2w_0(\xi)} . \tag{3.67}$$

Using (3.67) in (3.6) yields

$$A(X,T) = A[X_{c_0}(\xi),\xi]\left|\frac{2w_0(\xi)}{3w_0'(\xi)(T-\xi)-2w_0(\xi)}\right|^{1/2} \exp[-\gamma(T-\xi)]$$

and the matching condition, (3.3), then yields

$$A(X,T) = \frac{12w_0(\xi)}{\nu}\left|\frac{2w_0(\xi)}{3w_0'(\xi)(T-\xi)-2w_0'(\xi)}\right|^{1/2} \exp[-\gamma(T-\xi)] . \tag{3.68}$$

From equations (2.23) we have

$$\theta(X,T) = K[X-X_{c_0}(\xi)] - \Omega(T-\xi) + \theta[X_{c_0}(\xi),\xi] ,$$

on the characteristics, or from (2.32) and matching to the core

$$\theta(X,T) = K[X-X_0(\xi) - \lambda K^2(T-\xi)] . \tag{3.69}$$

Thus, from (3.68) and (3.69), the leading order singular solution for the leading tail is

$$U_0 = A \exp[-\frac{\theta}{\varepsilon}] . \tag{3.70}$$

From (3.48) we see that ξ_{max} satisfies the fundamental folding incomplete quadratic so that the matching solution in the leading tail, (3.43), becomes

$$\bar{u}_0 \sim \bar{D}[\frac{2\varepsilon^{1/3} d}{-2a\xi-b(T-T_1)}]^{1/2} \exp[-K^{(1)}(x-\lambda K^{(1)^2} t)]. \tag{3.71}$$

Matching (3.70) and (3.71) we obtain the condition

$$\bar{D} = \frac{12w_0(0)}{\nu}[\frac{ew_0(0)}{\varepsilon^{1/3}\pi d}]^{1/2} ,$$

so that

$$\bar{u}_0 = \frac{12}{\nu}[\frac{ew_0(0)d}{\varepsilon\pi}]^{1/2}\int_0^\infty \exp[\frac{d}{\varepsilon}(-\varepsilon^{2/3}Z\xi + \frac{1}{2}\varepsilon^{1/3}b\tau\xi^2 + \frac{1}{3}a\xi^3)]d\xi$$

$$\exp[-K^{(1)}(x-\lambda K^{(1)^2} t]. \tag{3.72}$$

Using (3.23) and the definitions of C_g, C, a and b we find from (3.65) that

$$C_g = 3w_0(0) ,$$

$$C = 0 ,$$

$$b = 3w_0'(0) = -4\gamma w_0(0) ,$$

and

$$a = 2\gamma w_0(0) \ .$$

Substituting these values into (3.54) and (3.55) we find the initial shock path and speed are

$$X_S = X_1 + 3w_0(0)(T-T_1) - \frac{3}{2}\gamma w_0(0)(T-T_1)^2,$$

and

$$\frac{dX_S}{dT} = 3w_0(0) - 3\gamma w_0(0)(T-T_1) \ .$$

CHAPTER IV

CUBIC FOLDING/CUSPED CAUSTICS

4.1 Introduction

In Chapter III we considered the case of focusing due to the moving core of a slowly varying solitary wave with first focusing occurring along the characteristic $\xi = 0$ and $T_f'(0) > 0$. We found for this case that the fundamental folding equation was quadratic in ξ (see Figure 3.7), and gave rise to a penumbral caustic (see Figure 3.4 and [20]). We now turn our attention to the other two possibilities: Section 4.2 gives the analysis for first focusing due to the moving core along a characteristic $\xi_1 > 0$ and $T_f'(\xi_1) = 0$, and Section 4.3 gives the analysis for first focusing due to initial conditions. In both of these cases we find that the fundamental folding equation is cubic in the characteristic variable (see Figure 4.2), and gives rise to a cusped caustic (see Figure 4.1).

4.2 First Focusing along $\xi_1 > 0$ and $T_f'(\xi_1) = 0$

In Section 3.2 we found that the characteristics for the leading tail due to the moving core were given by

$$X = F(T;\xi),\ F(\xi;\xi) = X_c(\xi),\quad \xi \geq 0\ , \tag{4.1}$$

where

$$\frac{dK}{dT} = -\lambda_X K^3\ , \tag{4.2}$$

and

$$\frac{dX}{dT} = 3\lambda K^2 . \tag{4.3}$$

For focusing we have

$$F_\xi(T_f;\xi) = 0 , \tag{4.4}$$

or

$$T_f = T_f(\xi) . \tag{4.5}$$

Thus the time, T_1, of first focusing is determined by minimizing T_f for $\xi \geq 0$. Chapter III analyzed the case where $T_1 = T_f(0)$ and $T_f'(0) > 0$. We now consider the case where

$$T_1 = T_f(\xi_1) , \quad \xi_1 > 0 , \tag{4.6}$$

and

$$T_f'(\xi_1) = 0 . \tag{4.7}$$

Using (4.7) in equation (3.19) we find that

$$F_{\xi\xi}(T_1;\xi_1) = 0 , \tag{4.8}$$

so that the fundamental folding equation, (3.18), must include cubic terms:

$$X-X_1-F_T(T_1;\xi_1)(T-T_1) - \frac{1}{2}F_{TT}(T_1;\xi_1)(T-T_1)^2$$

$$- \frac{1}{6} F_{TTT}(T_1;\xi_1)(T-T_1)^3 = F_{T\xi}(T_1;\xi_1)(T-T_1)(\xi-\xi_1)$$

$$+ \frac{1}{2} F_{TT\xi}(T_1;\xi_1)(T-T_1)^2(\xi-\xi_1)$$

$$+ \frac{1}{2} F_{T\xi\xi}(T_1;\xi_1)(T-T_1)(\xi-\xi_1)^2 + \frac{1}{6} F_{\xi\xi\xi}(T_1;\xi_1)(\xi-\xi_1)^3 .$$

The change of variable

$$\xi - \xi_1 = |T-T_1|^{1/2} f(S) , \tag{4.9}$$

then yields the similarity variable

$$S \equiv [X-X_1-C_g(T-T_1)]/|T-T_1|^{3/2} , \tag{4.10}$$

and the fundamental folding cubic is

$$af^3(S) + bf(S) = S . \tag{4.11}$$

For this case we have

$$C_g = F_T(T_1;\xi_1) ,$$

$$a = \frac{1}{6} F_{\xi\xi\xi}(T_1;\xi_1) ,$$

and

$$b = F_{T\xi}(T_1;\xi_1).$$

By differentiating equation (3.19) with respect to ξ, we find that $a < 0$ and $b > 0$. From equation (4.11) we find that there is one real root, and thus from (4.9) one characteristic value at each point (X,T) if

$$(\frac{S}{2a})^2 + (\frac{b}{3a})^3 > 0 ,$$

two distinct real roots (one a double root), and thus two characteristic values at each point (X,T) if

$$(\frac{S}{2a})^2 + (\frac{b}{3a})^3 = 0 ,$$

and three distinct real roots, and thus three characteristic values at each point (X,T) if

$$(\frac{S}{2a})^2 + (\frac{b}{3a})^3 < 0 .$$

Therefore, using (4.10), the cusped caustic is given by

$$X = X_1 + C_g(T-T_1) \pm 2a[\frac{-b}{3a}(T-T_1)]^{3/2} . \tag{4.12}$$

Figure 4.1 shows this cusped caustic and the number of characteristics in each region due to the moving core, while Figure 4.2 gives the evolution of the number of characteristics at each X due to the moving core.

Proceeding as in Section 3.3, but with the new similarity variable, (4.10), we find that

$$K = \bar{K}(T_1;\xi_1) + \bar{K}_T(T_1;\xi_1)(T-T_1) + \bar{K}_\xi(T_1,\xi_1)f(S)|T-T_1|^{1/2} + \ldots ,$$

and

$$A \sim \bar{A}(\xi)\left|[b-3af^2(S)](T-T_1)\right|^{-1/2} . \tag{4.13}$$

Using these expressions in (3.33) we then find that

$$B \quad |T-T_1|^{-5/2}\ \bar{f}(S) ,$$

and continuing to higher orders shows that the expansion for U breaks down when

$$T-T_1 = O(\varepsilon^{1/2}) .$$

Our new scales are thus

$$\tau = \varepsilon^{-1/2}(T-T_1) = \varepsilon^{1/2}(t-t_1) , \tag{4.14}$$

and

$$Z = \varepsilon^{-3/4}[X-X_1-C_g(T-T_1)] = \varepsilon^{1/4}[x-x_1-C_g(t-t_1)]. \tag{4.15}$$

Now seeking a solution of the form (3.38) we have

$$u_t = [\lambda^{(1)}K^{(1)^3}\phi - \varepsilon^{1/4}C_g\phi_Z + \varepsilon^{1/2}\phi_\tau]$$

$$\exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)] ,$$

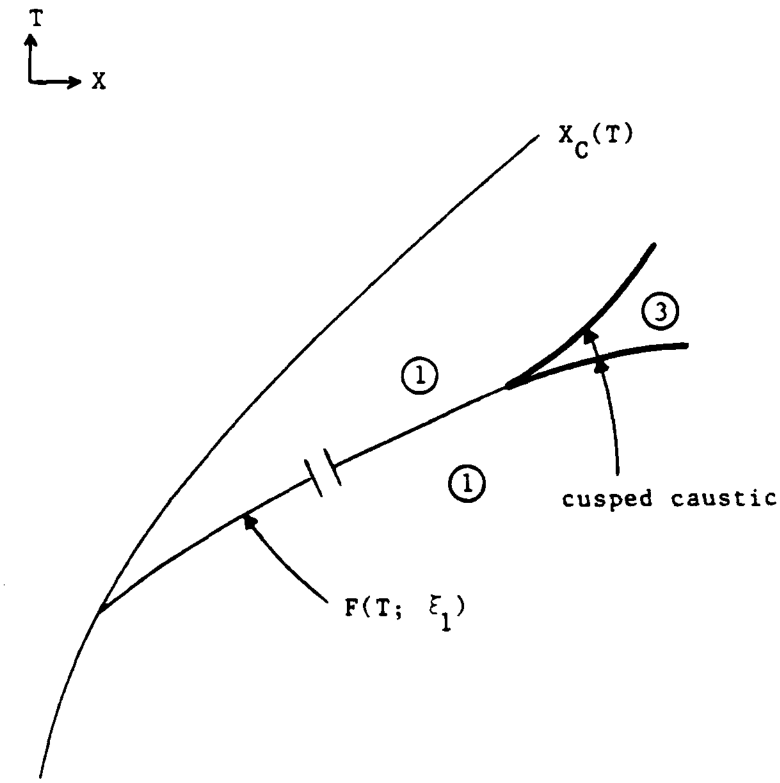

Figure 4.1 Cubic folding, cusped caustic, and the number of characteristics at each X due to the moving core.

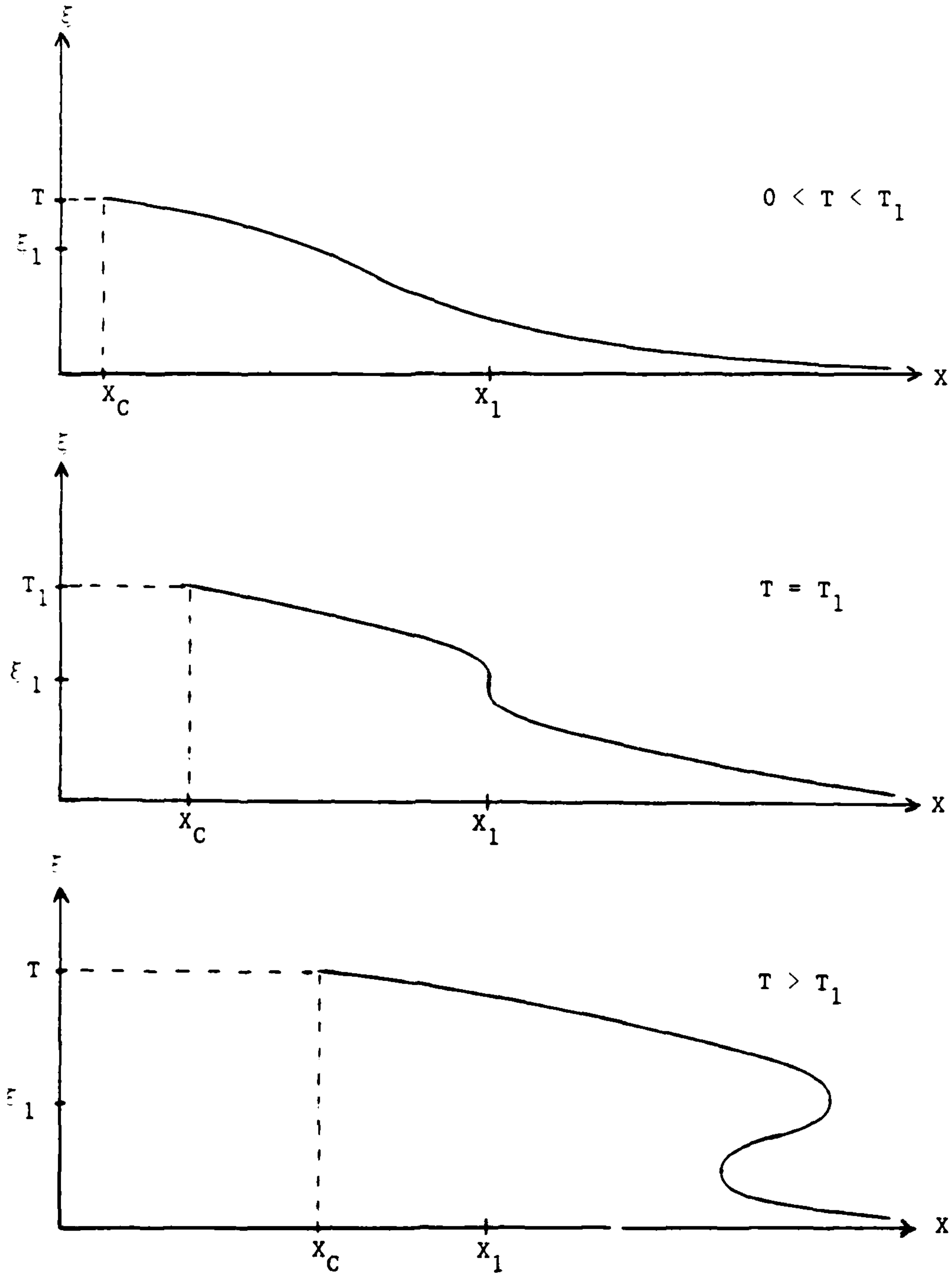

Figure 4.2 Evolution of the number of characteristics at each X due to the moving core.

and

$$u_{xxx} = [-K^{(1)^3}\phi + 3\varepsilon^{1/4}K^{(1)^2}\phi_Z - 3\varepsilon^{1/2}K^{(1)}\phi_{ZZ} + \varepsilon^{3/4}\phi_{ZZZ}]$$

$$\exp[-K^{(1)}(x-\lambda^{(1)}K^{(1)^2}t)] .$$

Thus from (2.21) we have

$$\phi_\tau = 3\lambda^{(1)}K^{(1)}\phi_{ZZ} - \varepsilon^{1/4}\phi_{ZZZ}$$

$$+ \varepsilon^{1/2}\{[\lambda_X^{(1)}(x-x_1) + \lambda_T^{(1)}(t-t_1)]K^{(1)^3}\phi - \gamma^{(1)}\phi\}$$

$$+ \text{H.O.T.}$$

As in Chapter III, to leading order we have the diffusion equation (3.40)

$$\phi_{0_\tau} = 3\lambda^{(1)}K^{(1)}\phi_{0ZZ} ,$$

with the solution (3.41)

$$\phi_0 = \int_{-\infty}^{\infty} D(\alpha)\exp[-\alpha Z + 3\lambda^{(1)}K^{(1)}\alpha^2\tau]d\alpha .$$

For the case we are now considering, we find from matching that $D(\alpha)$ is different from the previous case and

$$D(\alpha) = \begin{cases} 0 , & < \dfrac{-\varepsilon^{-1/4}b\xi_1}{6\lambda^{(1)}K^{(1)}} \\[2ex] \bar{D}\exp[54a(\dfrac{\lambda^{(1)}K^{(1)}}{b})^3\alpha^4] , & > \dfrac{-\varepsilon^{-1/4}b\xi_1}{6\lambda^{(1)}K^{(1)}} \end{cases}$$

The substitution

$$\alpha = \frac{\varepsilon^{-1/4}b}{6\lambda^{(1)}K^{(1)}}(\xi-\xi_1) ,$$

into (3.41) now yields

$$\phi_0 = \varepsilon^{-1/4} d\, \bar{D} \int_0^\infty \exp[\tfrac{d}{\ }\, h(\xi)]d \ , \tag{4.16}$$

where

$$d \equiv \frac{b}{6\lambda^{(1)}K^{(1)}} \ ,$$

and

$$h(\xi) \equiv \frac{1}{4}\, a(\xi-\xi_1)^4 + \frac{1}{2}\, b(T-T_1)(\xi-\xi_1)^2$$

$$- [X-X_1-C_g(T-T_1)](\xi-\xi_1) \ .$$

The integral, (4.16), with quartic phase, is similar in spirit to the oscillatory Pearcey integral which arises for cusped caustics in diffraction (see [9]). Laplace's method for the asymptotics of this exponential integral shows that the critical points of the quartic phase determined by

$$h'(\xi_c) = a(\xi_c-\xi_1)^3 + b(T-T_1)(\xi_c-\xi_1) - [X-X_1-C_g(T-T_1)] = 0 \ ,$$

satisfy the fundamental folding cubic (4.11). We see that there are one, two, or three real distinct critical points which correspond to a maximum, a maximum and a double point of inflection, or two relative maximums and a relative minimum of the quartic phase. Figure 4.3 illustrates the evolution of the quartic phase in the region near first focusing (the cusp of the caustic). The asymptotics of (4.16) are determined as in Chapter III. We find that the asymptotic solution is a single, slowly varying, exponentially decaying tail everywhere except at or near the points where $h(\xi)$ has two maximums of the same magnitude. The points where this occurs yield the initiation of a "shock path" as

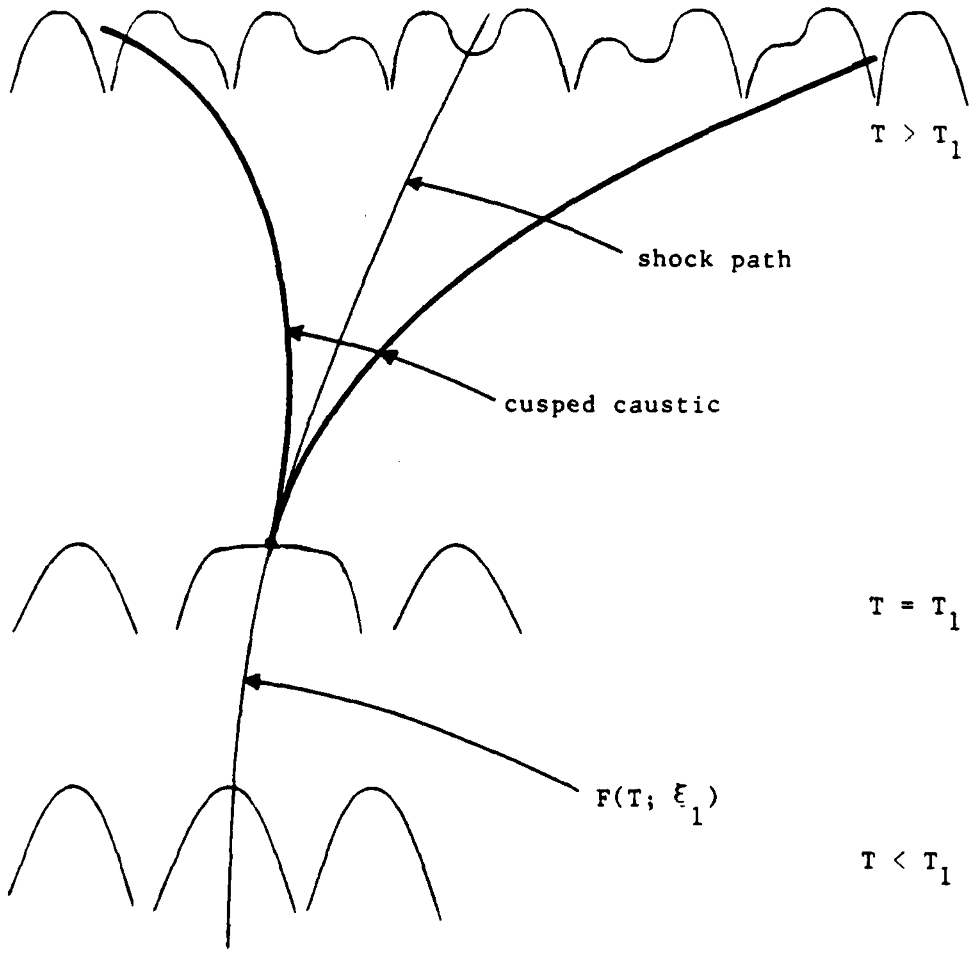

Figure 4.3 Evolution of the quartic phase near first focusing.

in the last chapter, where the asymptotic solution is the sum of two slowly vary tails and is in transition from one to the other. After the wave number shock forms, the shock path is determined by the intersection of two characteristics. For a while both of these characteristics initiate from the moving core ($\xi \geq 0$). However, if the characteristic on the right ($\xi < \xi_1$) tends to zero, then afterwards the shock path is determined by the intersection of two characteristics, one from the moving core ($\xi > \xi_1$) and one from the initial tail ($\zeta > 0$).

We have now considered two of three possible cases for focusing due to the moving core. In Chapter III we showed the analysis for first focusing along the characteristic $\xi = 0$, with $T_f'(0) > 0$, which led to a fundamental folding incomplete quadratic and a penumbral caustic. We have just shown the analysis for first focusing along the characteristic $\xi = \xi_1 > 0$, with $T_f'(\xi_1) = 0$, which led to a fundamental folding cubic and a cusped caustic. The final possibility for focusing due to the moving core is where first focusing occurs along the characteristic $\xi = 0$, with $T_f'(0) = 0$. The analysis for this case yields a fundamental folding incomplete cubic (due to $T_f'(0) = 0$, see Section 3.3), but the caustic is penumbral because we have $\xi \geq 0$ and we would need $\xi < 0$ to generate a cusp. Thus, cutting off the characeristic values at = 0 is equivalent, in this case, to cutting off one side of the cusp. This case of an incomplete cubic can be easily analyzed by our earlier methods. We do not include these details since we judge this case to be of little physical interest.

4.3 Focusing Due to Initial Conditions

In Section 3.2 we found that the characteristics for the leading tail due to initial conditions were given by

$$X = G(T;\zeta), \; G(0;\zeta) = \zeta, \quad \zeta > 0 \; , \tag{4.18}$$

where, as before, the method of characteristics yields

$$\frac{dK}{dT} = -\lambda_X K^3,$$

and

$$\frac{dX}{dT} = 3\lambda K^2 \; .$$

From equation (4.18) we see that focusing occurs for

$$G_\zeta(T_f;\zeta) = 0 \; , \tag{4.19}$$

or at times

$$T_f = T_f(\zeta) \; . \tag{4.20}$$

The remaining analysis for this case is identical to that in Section 4.2 for the previous case of focusing. The only differences in the two cases are seen by comparing Figures 4.1 and 4.2 with Figures 4.4 and 4.5 respectively, which show that the characteristic leading to first focusing, ζ_1, eminates from the initial conditions instead of the moving core, and the evolution of the number of characteristics at each X is slightly changed since all of the characteristics originate at time $T = 0$.

For a specialized treatment of focusing in the leading tail due to initial conditions (where slow variations leading to focusing are introuced strictly through slowly varying initial conditions and not by slowly varying coefficients) the reader is referred to [8].

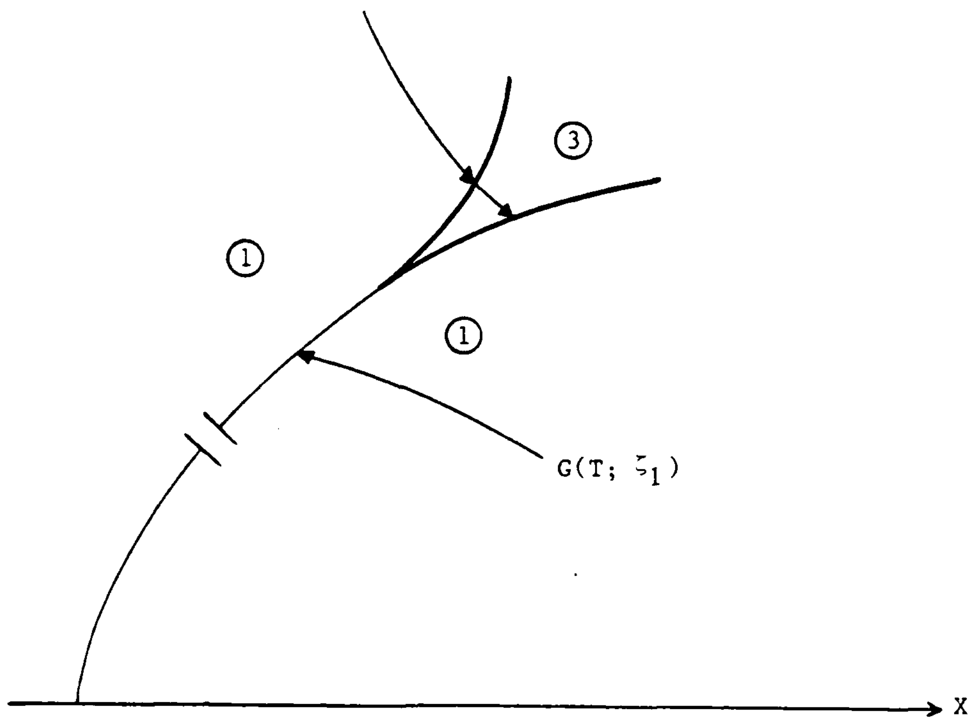

Figure 4.4 Cubic folding, cusped caustic, and the number of characteristics at each X due to the initial conditions.

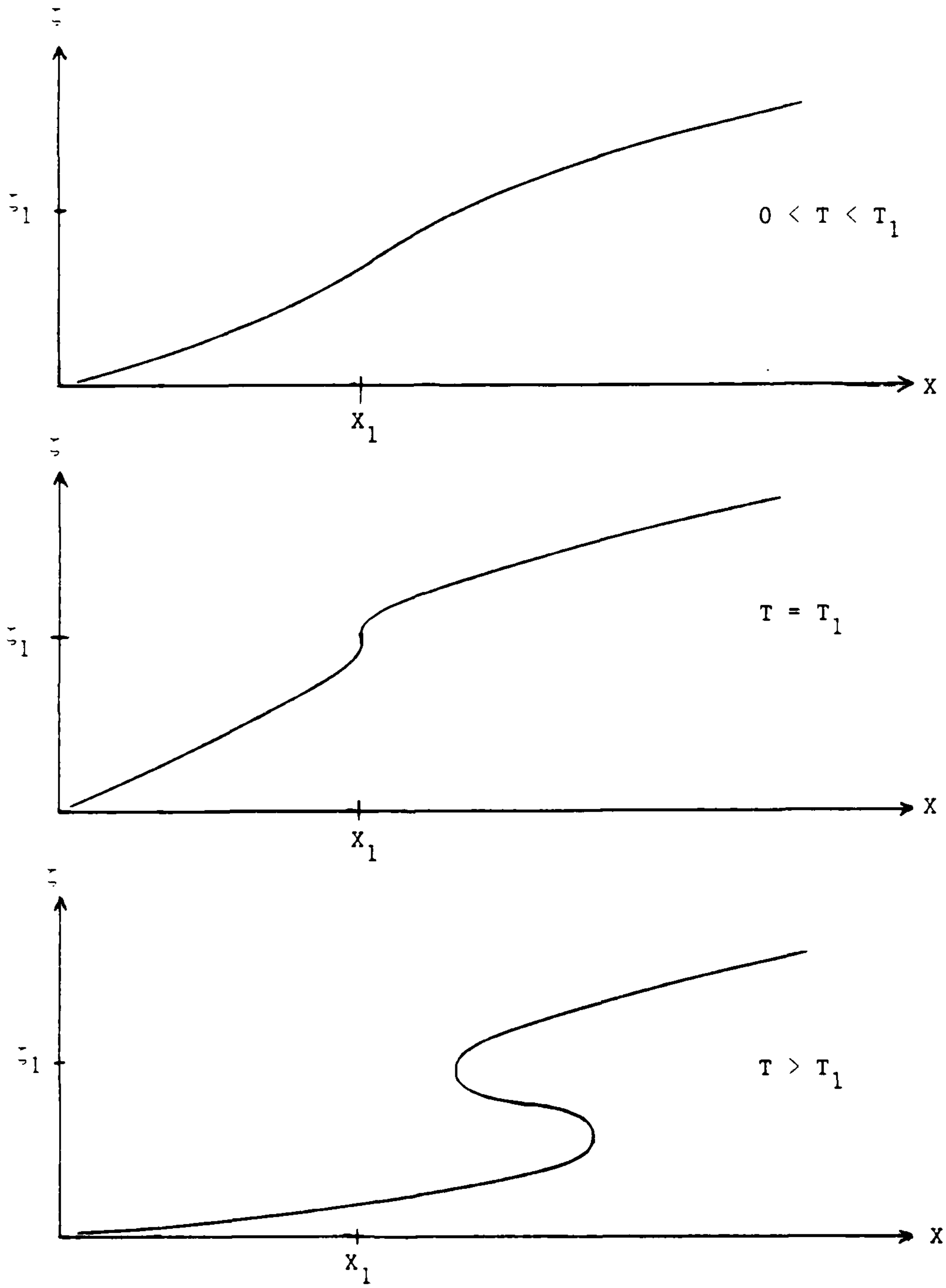

Figure 4.5 Evolution of the number of characteristics at each X due to initial conditions.

APPENDIX

Matching Solution of the Diffusion Equation Obtained from the Fundamental Solution

We have obtained a rather unusual solution, (3.41), of the diffusion equation, (3.40). Here, we will show that there are initial conditions which will yield this solution from the fundamental solution of the diffusion equation:

$$ {}_{0}(Z,\ ) = \frac{1}{\sqrt{4\pi(3\lambda^{(1)}K^{(1)}\tau}} \int_{-\infty}^{\infty} \phi_0(\beta,0) $$

$$ \exp\left[\frac{-(Z-\beta)^2}{4(3\lambda^{(1)}K^{(1)})\tau}\right] d\beta \quad . \qquad \text{(A.1)} $$

We take the initial condition, $\phi_0(\beta,0)$, from the matching solution, equation (3.41), which we have derived:

$$ \phi_0(\beta,0) = \bar{D}\int_0^{\infty}\exp[-\alpha\beta-\mu\alpha^3]d\alpha \quad , \qquad \text{(A.2)} $$

where $\mu = -12a(\frac{\lambda^{(1)}k^{(1)}}{b})^2$.

Substituting (A.2) into (A.1) and interchanging the order of integration we obtain

$$ \phi_0(Z,\tau) = \bar{D}\int_0^{\infty}Q(Z,\tau)\exp[-\mu\alpha^3]d\alpha, \qquad \text{(A.3)} $$

where

$$Q(Z,\tau) = \frac{1}{\sqrt{4\pi(3\lambda^{(1)}K^{(1)})\tau}} \int_{-\infty}^{\infty}\left\{\exp\left[-\alpha\beta\right]\right\}$$

$$\exp\left[\frac{-(Z-\beta)^2}{4(3\lambda^{(1)}K^{(1)})\tau}\right] d\beta . \qquad (A.4)$$

However, (A.4) is the fundamental solution of the following initial value problem:

$$Q_\tau = 3\lambda^{(1)}K^{(1)}Q_{ZZ} \quad ,$$

$$Q(Z,0) = \exp[-\alpha Z].$$

Thus, we solve for Q and obtain

$$Q(Z,\tau) = \exp[-\alpha Z + 3\lambda^{(1)}K^{(1)}\alpha^2\tau] . \qquad (A.5)$$

Substituting (A.5) into (A.3) then yields the solution of the diffusion equation which we have obtained by matching

$$\phi_0(Z,\tau) = \bar{D}\int_0^{\infty}\exp[-\alpha Z + 3\lambda^{(1)}K^{(1)}\alpha^2\tau - \mu\alpha^3]d\alpha . \qquad (A.6)$$

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